Physics 464/564

Homework #7 Due June 4th, 2009

1. The gradient ∇, divergence $\nabla \bullet$, and rotational $\nabla \mathbf{x}$" operators were defined in Lecture 8.

In the following questions $\boldsymbol{r}$ = (x, y, z) stands for the vector position,

and $\boldsymbol{k}$ = (k_x, k_y, k_z) is a constant vector,

1.1 For the scalar function $f(\boldsymbol{r}) = e^{i\boldsymbol{k} \bullet \boldsymbol{r}}$, evaluate ∇f .

1.2 For the vector field $\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E_o} e^{i\boldsymbol{k} \bullet \boldsymbol{r}}$, where $\boldsymbol{E_o}$ = (E_{ox}, E_{oy}, E_{oz}) is a constant vector,

1.2.a Apply the divergence operator $\nabla \bullet$ to the field $\boldsymbol{E}$ and show that $\nabla \bullet \boldsymbol{E} = i\boldsymbol{k} \bullet \boldsymbol{E}$

(the right side stands for the scalar product between $\boldsymbol{k}$ and $\boldsymbol{E}$.

1.2.b Apply the rotational operator $\nabla \mathbf{x}$ to the field $\boldsymbol{E}$ and show that $\nabla \mathbf{x}\, \boldsymbol{E} = i\boldsymbol{k} \mathbf{x}\, \boldsymbol{E}$

(the right side stands for the vector product between $\boldsymbol{k}$ and $\boldsymbol{E}$.

2. The Maxwell Equations(ME) in the free space are given by

$$\nabla \bullet \mathbf{E} = \mathbf{0} \qquad \nabla \mathrm{x} \mathbf{E} + \frac{\partial}{\partial t} \mathbf{B} = 0$$

$$\nabla \mathrm{x} \mathbf{B} - \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \mathbf{E} = \mathbf{0} \qquad \nabla \bullet \mathbf{B} = 0 \ ,$$

Consider electromagnetic waves of the form,

$$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E_o} e^{i(\boldsymbol{k} \bullet \boldsymbol{r} - \omega t)} \quad \text{and} \quad \boldsymbol{B}(\boldsymbol{r}) = \boldsymbol{B_o} e^{i(\boldsymbol{k} \bullet \boldsymbol{r} - \omega t)}$$

where ω is assumed to be known. $\boldsymbol{E_o}$, $\boldsymbol{B_o}$ and $\boldsymbol{k}$ are (unknown) constant vectors.

2.1 Use the ME to show that $\boldsymbol{k} \bullet \boldsymbol{E_o} = 0$ and $\boldsymbol{k} \bullet \boldsymbol{B_o} = 0$
That is, the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are perpendicular to the propagation vector $\boldsymbol{k}$.
Hint: Use the ME involving the divergence operator

2.2 Use the ME to show that $\boldsymbol{E}$, and $\boldsymbol{B}$ are perpendicular to each other.
Hint: Use the ME involving the rotational operator.

3. In a dielectric medium of polarization **P**, where there are not extra free charges, except the ones bounded to the atoms, the ME take the form

$$\nabla \bullet (\varepsilon_0 \mathbf{E} + \mathbf{P}) = 0 \qquad \nabla \mathrm{x} \mathbf{E} + \frac{\partial}{\partial t}\mathbf{B} = 0$$

$$\nabla \mathrm{x}(\frac{1}{\mu_0}\mathbf{B}) = \frac{\partial}{\partial t}(\varepsilon_0 \mathbf{E} + \mathbf{P}) \qquad \nabla \bullet \mathbf{B} = 0$$

The behavior of electromagnetic waves traveling in such a medium is governed by the Eq .

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} = -\frac{1}{\varepsilon_0}\nabla(\nabla \bullet \mathbf{P}) + \frac{1}{\varepsilon_0 c^2}\frac{\partial^2}{\partial t^2}\mathbf{P}$$

For an isotropic medium $\boldsymbol{p} = \varepsilon_0 \chi(\omega)\boldsymbol{E}$, $\boldsymbol{P} = N\boldsymbol{p}$

where $\boldsymbol{p}$ is the molecular dipole, and **P** is the dipole moment per unit volume.

3.1 Consider a plane wave of the form

$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}_{\boldsymbol{o}} e^{i(\boldsymbol{k}\bullet\mathbf{r}-\omega t)}$ where $\boldsymbol{k}$ = (0,0,1)

Fid the corresponding polarization vector **P** in the medium caused by this incident plane wave.

3.2 For the particular Polarization vector **P** found in part 3.1 above, show that $\nabla \bullet \mathbf{P} = 0$

3.3 Based on the result found in 3.2 above, write down the corresponding equation for the field **E**.