A. La Rosa Lecture Notes

APPLIED OPTICS

Maxwell Equations for electromagnetic waves inside a medium

$$\nabla \bullet (\varepsilon_0 \mathbf{E} + \mathbf{P}) = \rho_{free} \qquad \nabla x \mathbf{E} + \frac{\partial}{\partial t} \mathbf{B} = 0$$

$$\nabla x (\frac{1}{\mu_0} \mathbf{B} - \mathbf{M}) = \mathbf{J}_{free} + \frac{\partial}{\partial t} (\varepsilon_0 \mathbf{E} + \mathbf{P}) \qquad \nabla \bullet \mathbf{B} = 0 \ ,$$

P is the polarization vector (dipole moment per unit volume)
P = N**p**, where **p** is the electrical dipole moment of one molecule
N is the number of molecules per unit volume

M is the polarization vector (dipole moment per unit volume)
M = N**m**, where **m** is the electrical dipole moment of one molecule
N is the number of molecules per unit volume

- Consider the particular case of non-magnetic material (**M = 0**)

$$\nabla \bullet (\varepsilon_0 \mathbf{E} + \mathbf{P}) = \rho_{free} \qquad \nabla x \mathbf{E} + \frac{\partial}{\partial t} \mathbf{B} = 0$$

$$\nabla x (\frac{1}{\mu_0} \mathbf{B}) = \mathbf{J}_{free} + \frac{\partial}{\partial t} (\varepsilon_0 \mathbf{E} + \mathbf{P}) \qquad \nabla \bullet \mathbf{B} = 0 \ ,$$

Further, let's focus our interest in dielectric materials, where there are not extra free charges, except the ones bounded to the atoms.

$$\nabla \bullet (\varepsilon_0 \mathbf{E} + \mathbf{P}) = 0 \qquad \nabla x \mathbf{E} + \frac{\partial}{\partial t} \mathbf{B} = 0$$

$$\nabla x (\frac{1}{\mu_0} \mathbf{B}) = \frac{\partial}{\partial t} (\varepsilon_0 \mathbf{E} + \mathbf{P}) \qquad \nabla \bullet \mathbf{B} = 0$$

The wave equation

To solve the equations above, let's take first the rotational of the Faraday's law

$$\nabla x \nabla x \mathbf{E} + \frac{\partial}{\partial t} \nabla x \, \mathbf{B} = 0$$

Using the identity $\nabla x \nabla x \mathbf{E} = \nabla(\nabla \bullet \mathbf{E}) - \nabla^2 \mathbf{E}$,

$$\nabla^2 \mathbf{E} = \nabla(\nabla \bullet \mathbf{E}) + \frac{\partial}{\partial t} \nabla \mathrm{x}\, \mathbf{B}$$

Using the modified Ampere's law $\nabla \mathrm{x}(\frac{1}{\mu_0}\mathbf{B}) = \frac{\partial}{\partial t}(\varepsilon_0 \mathbf{E} + \mathbf{P})$,

$$\nabla^2 \mathbf{E} = \nabla(\nabla \bullet \mathbf{E}) + \frac{\partial}{\partial t}\frac{\partial}{\partial t}(\mu_0 \varepsilon_0 \mathbf{E} + \mu_0 \mathbf{P})$$

Since $c^2 = 1/(\mu_0 \varepsilon_0)$

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} = \nabla(\nabla \bullet \mathbf{E}) + \frac{1}{\varepsilon_0 c^2}\frac{\partial^2}{\partial t^2}\mathbf{P}$$

Using the first Maxwell's Eq. $\nabla \bullet (\varepsilon_0 \mathbf{E} + \mathbf{P}) = 0$,

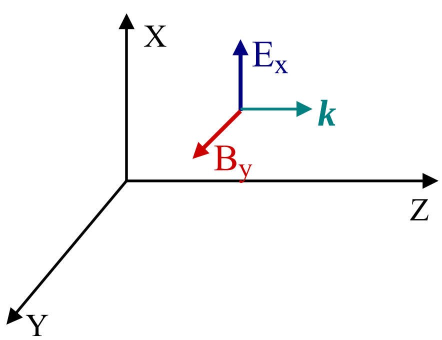
$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} = -\frac{1}{\varepsilon_0}\nabla(\nabla \bullet \mathbf{P}) + \frac{1}{\varepsilon_0 c^2}\frac{\partial^2}{\partial t^2}\mathbf{P} \qquad (1)$$

Notice, if **P** were zero, then the electromagnetic fields would travel at speed c. But in the presence of **P**, the situation is different.

Plane waves solutions

Things simplify if we consider isotropic dielectrics, i.e. **P** and **E** are always parallel to each other.

Let's consider plane waves $e^{i\mathbf{k}\bullet\mathbf{r}}$ as potential solutions. Without loosing generality, we can choose $\boldsymbol{k} = k\hat{\mathbf{z}}$ and that the wave is polarized in the x-direction.



$$\boldsymbol{E}(\boldsymbol{r},t) = E_o e^{i(\mathbf{k}\bullet\mathbf{r} - \omega t)}(1, 0, 0)$$
$$= E_o e^{i(\mathbf{k}\bullet\mathbf{r} - \omega t)}\hat{\mathbf{x}}$$
$$= E_o e^{i(k z - \omega t)}\hat{\mathbf{x}}$$

$$\boldsymbol{E}(\mathbf{r},t) = (E_x, 0, 0) \qquad (2)$$
$$E_x = E_o e^{i(k z - \omega t)}$$

which is a wave with phase velocity $v_{ph} = \omega / \kappa$. The index of refraction can be introduced here,

$$n \equiv c / v_{ph} = c\, \kappa \,/\, \omega \,. \qquad (3)$$

What we need to find out is what values of k are compatible with Eq. (1)

Since

$$\boldsymbol{p} = \varepsilon_0 \chi(\omega) \boldsymbol{E} \qquad (4)$$
$$\boldsymbol{P} = N\boldsymbol{p}$$

Notice that for the field $\boldsymbol{E}$ given in expression (2), one obtains $\nabla \cdot \mathbf{P} = 0$.
Thus,

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} = \frac{N\chi(\omega)}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} \quad \text{, and}$$

$$\nabla^2 \mathbf{E} - \frac{1 + N\chi(\omega)}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} = 0 \qquad (5)$$

Replacing (2) in (5)

$$k^2 - \frac{1 + N\chi(\omega)}{c^2}\omega^2 = 0$$

$$k^2 = \frac{\omega^2}{c^2}\left(1 + N\chi(\omega)\right) \qquad (6)$$

Comparing (3) and (6)

$$n^2 = 1 + N\chi(\omega) \qquad (7)$$

$$n^2 = 1 + N\frac{e^2/m\varepsilon_0}{\omega_o{}^2 - \omega^2 + j\gamma\omega}$$