Andres

The origin of the Refractive Index

Reference: Feynman Lectures vol 1, CH-30 & CH-31

A. An accelerated charge produce electromagnetic fields

B. Electric field produced by a plane-section of oscillating charges.

Evaluation of the integral $\int e^{-i\frac{r}{c}\omega}\,dr$

C Index of refraction

First approach: Form of the electric field if a slow-down of its speed is assumed

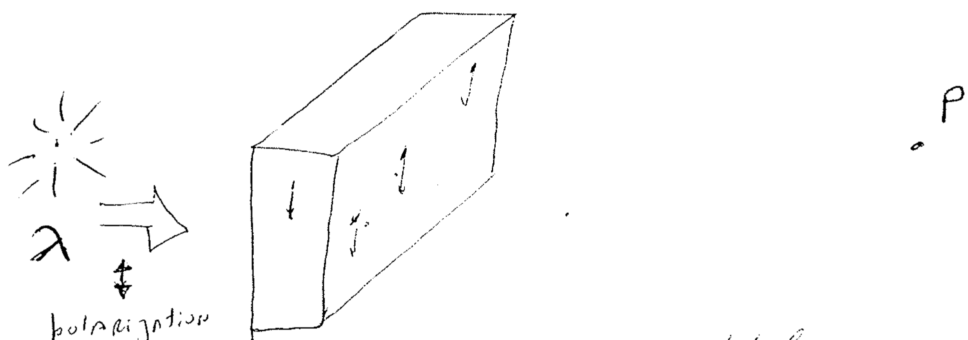
Second approach: Form of the electric field when the contribution ~~of~~ from accelerating charges are taken into account.

The origin of the Refractive Index

It is approximately true that light does appear to travel at a speed c/n through a material whose index of refraction is n,

But

the electromagnetic fields at P (i.e. light) are still produced by the motion of all the charges, including the charges moving in the material



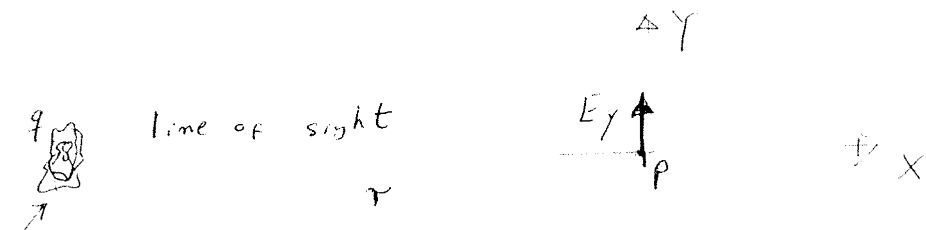
All these basic contributions to the total electromagnetic field at P (from the source and from the atoms in the glass) travel at the ultimate velocity c

Our objective is: to understand how the apparently slower velocity of light in the glass comes about.

Preliminary calculations:

The field of a plane of oscillating charges

A. An accelerated charge produce electromagnetic fields



trajectory of the charge q

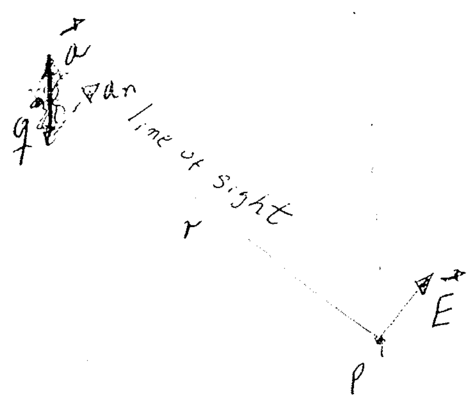
(1)

$$E_Y(P) = \frac{q}{4\pi\epsilon_0}\frac{1}{r}\,a_Y\left(t-\frac{r}{c}\right)$$

electric field at P at the time t

component of the acceleration of the charge, perpendicular to the line of sight

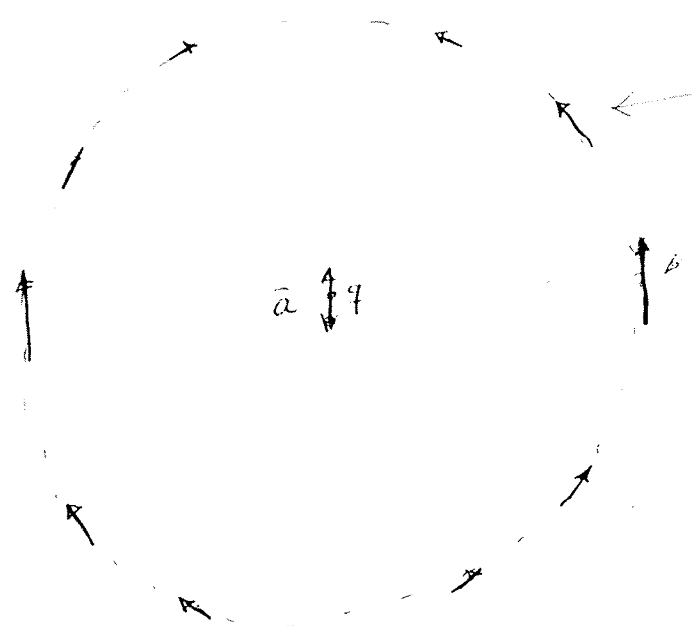
the acceleration is evaluated at a retarded time $t-\frac{r}{c}$ because the signal takes a time $\frac{r}{c}$ to arrive from the charge location to the point P



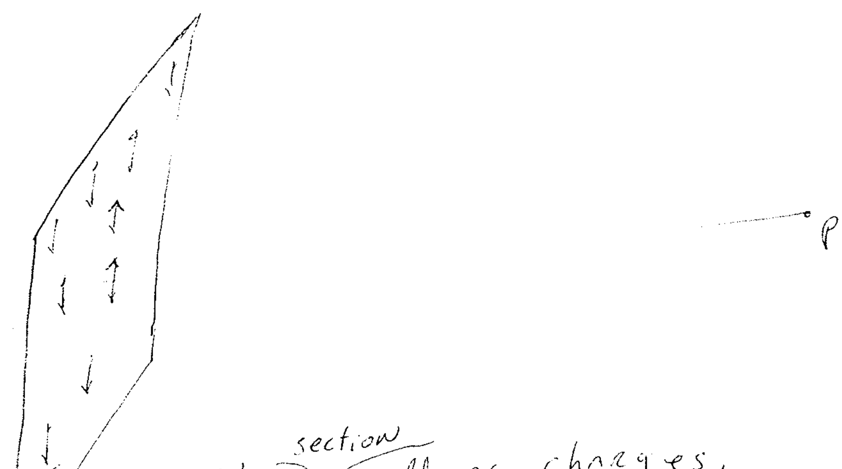
(2)

$$E(P) = \frac{q}{4\pi\epsilon_0}\frac{1}{r}\,a_n\left(t-\frac{r}{c}\right)$$

normal component of the charge's acceleration



B.- Electric field produced by a plane section of oscillating charges.



- Plane section full of charges,
- All charges oscillating synchronously, with the same amplitude x_0 and the same phase
- There are η charges per unit area of the plane,
- Each charge, having charge $= q$, moves around its own average position according to

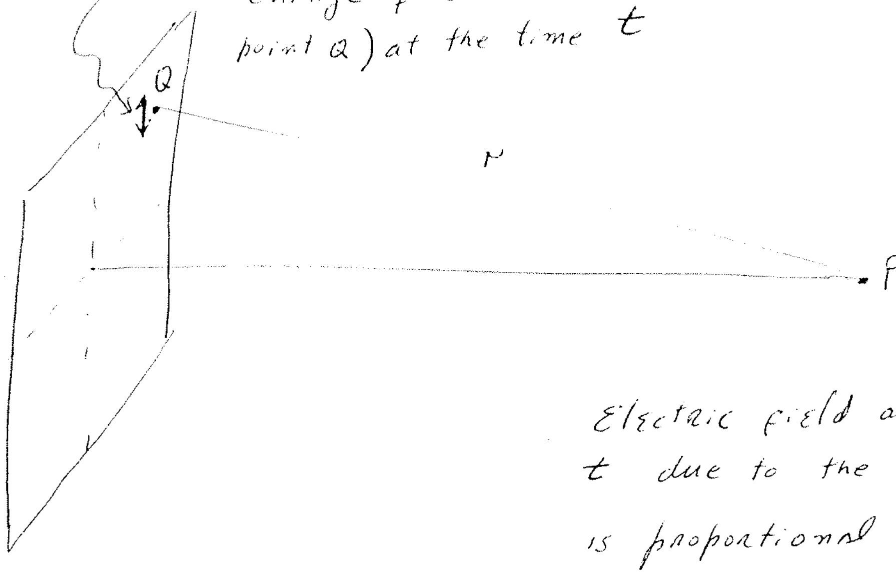
$$x = x_0 \cos \omega t$$

or

$$x = x_0 e^{i\omega t}$$

$a = -x_0 \omega^2 e^{i\omega t}$

acceleration of the charge q (located at point Q) at the time t



Electric field at point P and at the time t due to the charge q located at Q is proportional to the acceleration that the charge q had the earlier time $t - \frac{r}{c}$

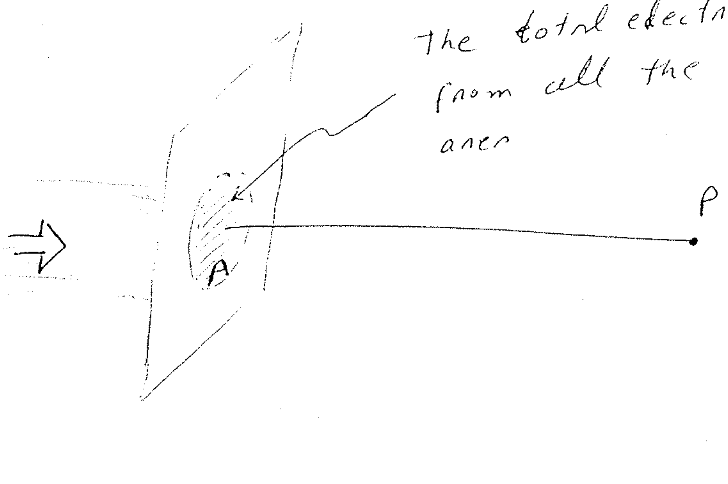
$$E\left(\substack{\text{at } P \text{ due to} \\ \text{charge at } Q}\right) \propto -x_0 \omega^2 e^{i\omega(t - \frac{r}{c})}$$

More exactly

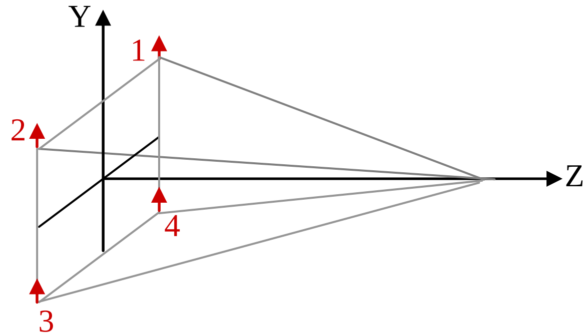
$$E\left(\substack{\text{at } P \text{ due to} \\ \text{charge at } Q}\right) = \left[\frac{q}{4\pi\epsilon_0 c^2} \frac{1}{r}\right] \omega^2 x_0 e^{i(\omega t - r/c)} = E(\text{at } P, t) \quad (3)$$

Let's consider that the oscillating charges are produced by a light beam incident, for example, in a plane cross section of glass

The total electric field is the contribution from all the charges in this cross section area



Consider 4 dipoles located at the corners

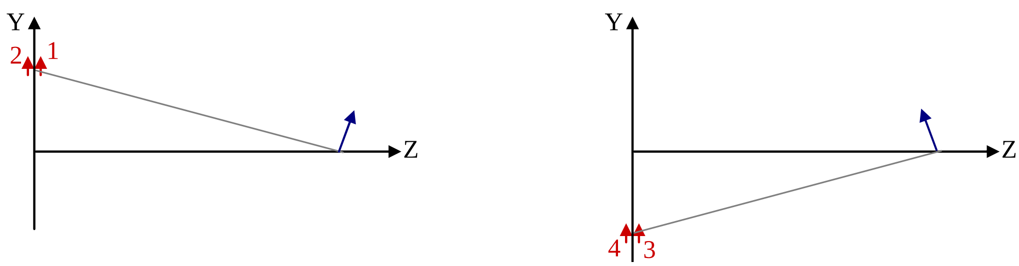


- Notice the 4 dipoles (oriented along the Y-axis) will produce a net electric field pointing in the z-axis.

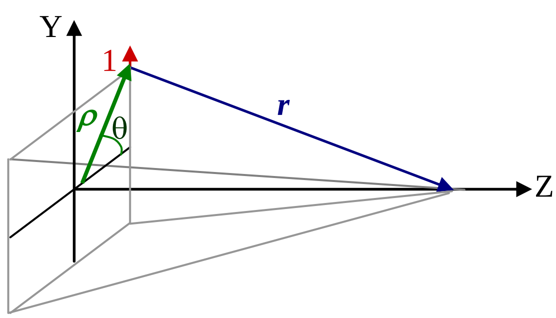
  Justification:

  The dipoles 1 and 2 produce a net field in the plane ZY

  Similarly, the dipoles 3 and 4 produce a net field in the plane ZY



- Thus, it will be enough to calculate just the y-component of a vector normal to r (see diagram below.



$\boldsymbol{r} = -\rho\ (\mathrm{Cos}\theta, \mathrm{Sin}\theta, 0)\ \ +\ \ z\ (0,0,1)$

$-\rho\ \boldsymbol{e}_\rho\ \ +\ \ z\ \boldsymbol{e}_z$

The electric field is normal to the line of sight $\boldsymbol{r}$. A unit vector normal to $\boldsymbol{r}$ and in the plane formed by $\boldsymbol{n}$ and $\boldsymbol{e}_y$ is given by,

$$\boldsymbol{n} = \frac{(\boldsymbol{r}\ \mathrm{x}\mathbf{e}_{\mathrm{y}})\ \mathrm{x}\,\boldsymbol{r}}{\left|(\boldsymbol{r}\ \mathrm{x}\mathbf{e}_{\mathrm{y}})\ \mathrm{x}\,\boldsymbol{r}\right|}$$

The electric field at (0.0,z) produced by the dipole "1" is along $\boldsymbol{n}$.

- $\boldsymbol{r} \times \boldsymbol{e}_y = -\rho\, \boldsymbol{e}_\rho \times \boldsymbol{e}_y + z\, \boldsymbol{e}_z \times \boldsymbol{e}_y$
  $= -\rho\,(0, 0, \mathrm{Cos}\theta) - z\,(1, 0, 0) = -\rho\, \boldsymbol{e}_z - z\, \boldsymbol{e}_x$

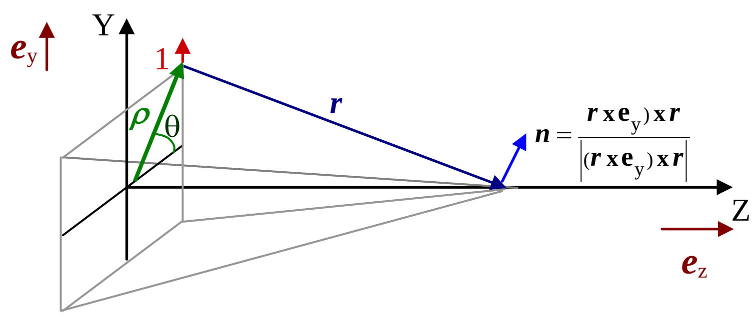
$$(\boldsymbol{r} \times \boldsymbol{e}_y) \times \boldsymbol{r} = (-\rho\, \boldsymbol{e}_z - z\, \boldsymbol{e}_x) \times (-\rho\, \boldsymbol{e}_\rho + z\, \boldsymbol{e}_z)$$
$$= \rho^2\, \boldsymbol{e}_z \times \boldsymbol{e}_\rho + z\,\rho\, \boldsymbol{e}_x \times \boldsymbol{e}_\rho + z^2\, \boldsymbol{e}_y$$
$$= \rho^2\, (0,0,1) \times (\mathrm{Cos}\theta, \mathrm{Sin}\theta, 0) + z\,\rho\,(1,0,0) \times (\mathrm{Cos}\theta, \mathrm{Sin}\theta, 0) + z^2\, \boldsymbol{e}_y$$
$$= \rho^2\, (-\mathrm{Sin}\theta, \mathrm{Cos}\theta, 0) + z\,\rho\,(0, 0, \mathrm{Sin}\theta) + z^2\,(0, 1, 0)$$

$$(\boldsymbol{r} \times \boldsymbol{e}_y) \times \boldsymbol{r} = (-\rho^2\, \mathrm{Sin}\theta, \rho^2\, \mathrm{Cos}\theta, 0) + (0, 0, z\,\rho\, \mathrm{Sin}\theta) + (0, z^2, 0)$$

$$(\boldsymbol{r} \times \boldsymbol{e}_y) \times \boldsymbol{r} = (-\rho^2\, \mathrm{Sin}\theta,\ \ z^2 + \rho^2\, \mathrm{Cos}\theta,\ \ z\,\rho\, \mathrm{Sin}\theta)$$

$$|(\boldsymbol{r} \times \boldsymbol{e}_y) \times \boldsymbol{r}|^2 = \rho^4 + z^4 + 2\, z^2 \rho^2\, \mathrm{Cos}\theta + z^2 \rho^2\, \mathrm{Sin}^2\theta$$

$$|(\boldsymbol{r} \times \boldsymbol{e}_y) \times \boldsymbol{r}|^2 = \rho^4 + z^4 + z^2 \rho^2\, (2\mathrm{Cos}\theta + \mathrm{Sin}^2\theta)$$



$$\boldsymbol{n} = \frac{\boldsymbol{r}\ \mathrm{x}\mathbf{e}_{\mathrm{y}})\ \mathrm{x}\,\boldsymbol{r}}{\left|(\boldsymbol{r}\ \mathrm{x}\mathbf{e}_{\mathrm{y}})\ \mathrm{x}\,\boldsymbol{r}\right|}$$

$$\boldsymbol{n} = \frac{(-\rho^2 Sin\theta,\ \ z^2 + \rho^2 Cos\theta,\ \ z\rho Sin\theta)}{\left|(\boldsymbol{r}\ \mathrm{x}\mathbf{e}_{\mathrm{y}})\ \mathrm{x}\,\boldsymbol{r}\right|}$$

The component of n along the y-axis is given by,

$$\boldsymbol{n} \bullet \mathbf{e}_{\mathrm{y}} = \frac{z^2 + \rho^2 Cos\theta}{\left|(\boldsymbol{r}\ \mathrm{x}\mathbf{e}_{\mathrm{y}})\ \mathrm{x}\,\boldsymbol{r}\right|}$$

$$\boldsymbol{n} \bullet \mathbf{e}_{y} = \frac{z^2 + \rho^2 Cos\theta}{\left[\rho^4 + z^4 + z^2 \rho^2 (2Cos\theta + Sin^2\theta)\right]^{1/2}}$$

For point along the x-axis ($\theta = 0$),

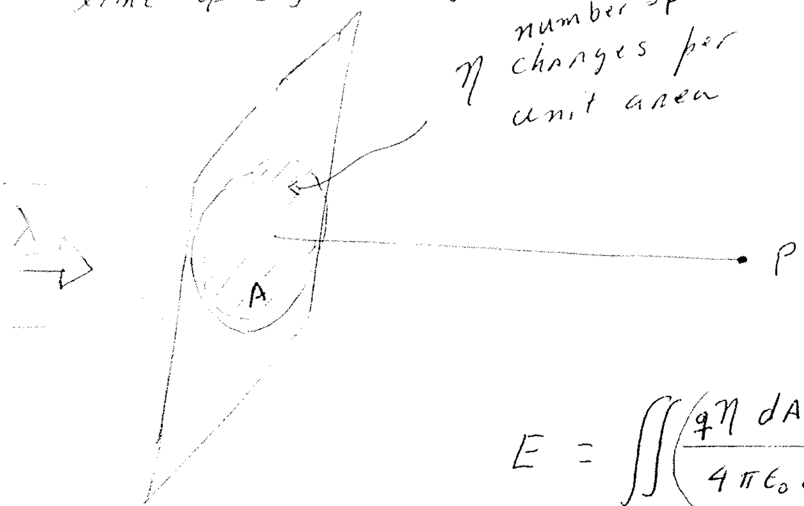
$$\boldsymbol{n} \bullet \mathbf{e}_{y} = 1$$

For point along the y-axis ($\theta = 90^o$),

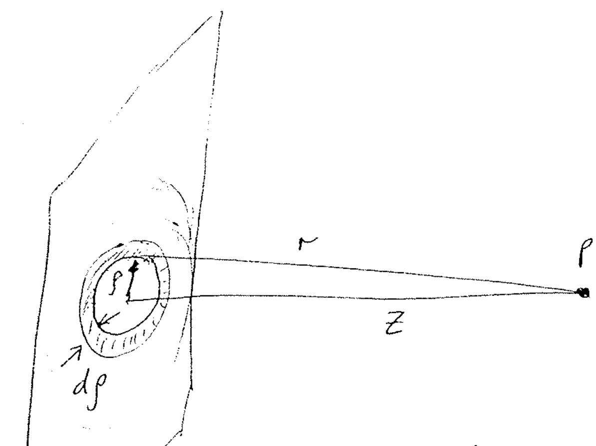
$$\boldsymbol{n} \bullet \mathbf{e}_{y} = \frac{z^2}{\left[\rho^4 + z^4 + z^2 \rho^2 \right]^{1/2}} \xrightarrow[\rho \to \infty]{} \frac{z^2}{\rho^2}$$

Thus, in the derivation made by Feynman, who takes $\boldsymbol{n} \bullet \mathbf{e}_{y} = 1$, the approximation is good except for points near the Y axis. The farther the point is from the origin, the lesser its contribution to the total electric field.

Since the cross section area A is assumed to be far away from the observation point P, the omission of ~~the~~ considering the component of the acceleration normal to the line of sight is justified.



$$E = \iint_A \left(\frac{q\eta \, dA}{4\pi\epsilon_0 c^2} \frac{1}{r} \right) \omega^2 x_0 \, e^{i(\omega t - r/c)}$$



$$dA = 2\pi\rho \, d\rho$$

$$E(\text{at } P) = \int_{\rho=0}^{\infty} \left(\frac{q\eta \, 2\pi\rho \, d\rho}{4\pi\epsilon_0 c^2} \frac{1}{r} \right) \omega^2 x_0 \, e^{i(\omega t - \frac{r}{c})}$$

Notice $\rho^2 + z^2 = r^2$

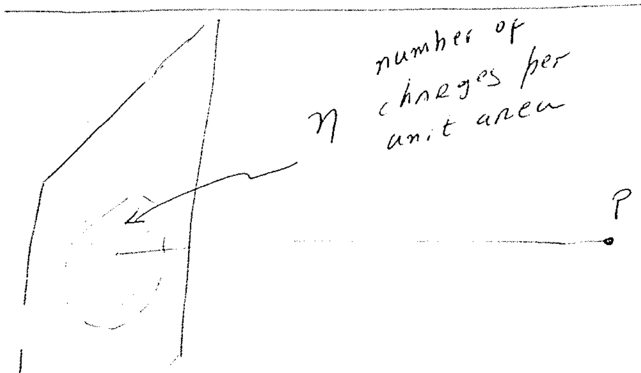
$$2\rho \, d\rho = 2r \, dr$$

$$E(\text{at } P) = \int_{r=z}^{\infty} \left(\frac{q\eta \, r \, dr}{2\epsilon_0 c^2} \frac{1}{r} \right) \omega^2 x_0 \, e^{i(\omega t - r/c)}$$

$$E\,(\text{at } P) = \frac{q\,\omega^2 x_0}{2\,\epsilon_0 c^2} \int_{r=z}^{\infty} \eta\, dr\, e^{i(\omega t - r/c)}$$

$$= \frac{q\,\omega^2 x_0\, e^{i\omega t}}{2\,\epsilon_0 c^2} \int_{r=z}^{\infty} e^{-i\frac{r}{c}\omega}\, \eta\, dr \qquad (4)$$

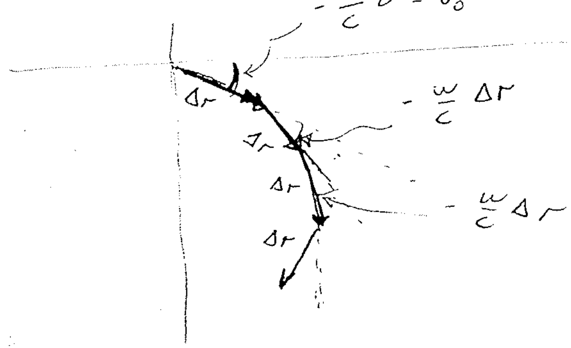
Evaluation of the integral $\int_{r=z}^{\infty} e^{-i\frac{r}{c}\omega}\, \eta\, dr$

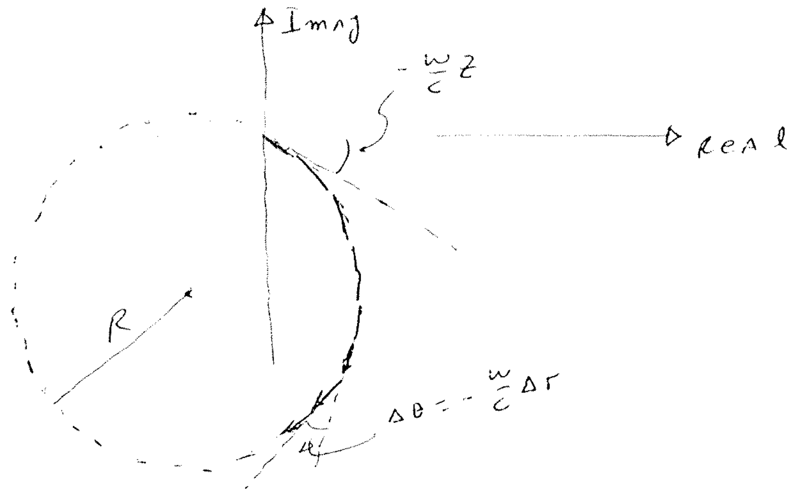


Let's consider first that η is constant

then, the integral $\int_{r=z}^{\infty} e^{-i\frac{\omega}{c}r}\, dr$ can be interpreted

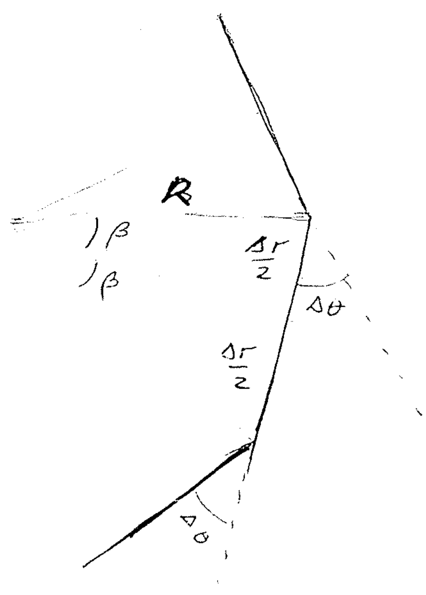
as: the sum of many small complex numbers, each of magnitude dr and with an angle that increases by $-\frac{\omega}{c}\,dr = \Delta\theta$, starting from $\theta_0 = -\frac{\omega}{c}z$





As we keep adding the little complex numers $\Delta r e^{-i\frac{w}{c}\Delta r}$ a circunference of radius R is obtained.

Let's calculate R



$$R \sin\beta = \frac{\Delta r}{2}$$

$$R = \frac{\Delta r/2}{\sin\beta}$$

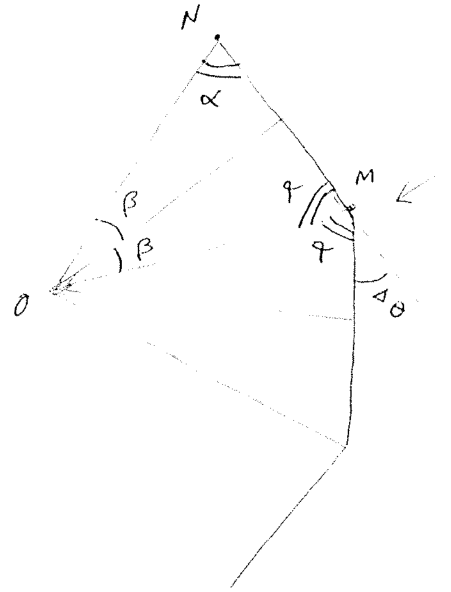
But, what is the value of β?

Answer: $2\beta = \Delta\theta$

Accordingly,

$$R = \frac{\Delta r/2}{\sin\left(\frac{\Delta\theta}{2}\right)}$$

In ΔONM:
$2\beta + 2\alpha = \pi$

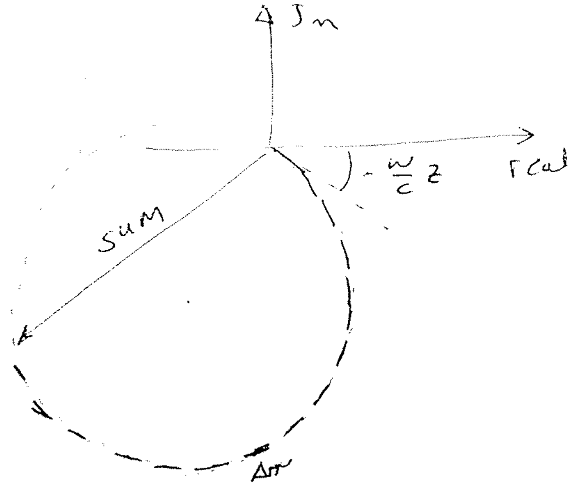


$2\alpha + \Delta\theta = \pi$

$\Rightarrow 2\beta = \Delta\theta$

For small $\Delta\theta$ (small $\frac{\omega}{c}\Delta r$ or $\Delta r \ll \frac{c}{\omega} = \frac{\lambda}{2\pi}$)

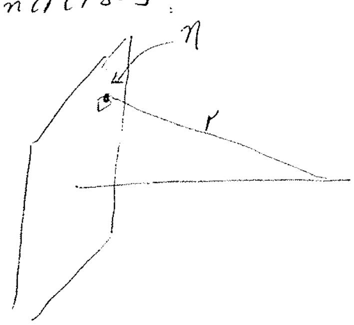
$$R = \frac{\Delta r/2}{\sin\frac{\Delta\theta}{2}} \approx \frac{\Delta r/2}{\Delta\theta/2} = \frac{\Delta r}{\frac{\omega}{c}\Delta r} \Rightarrow \boxed{R = \frac{c}{\omega}} = \frac{\lambda}{2\pi}$$

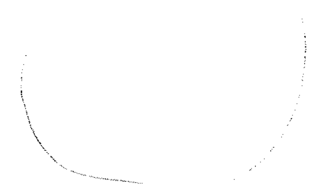


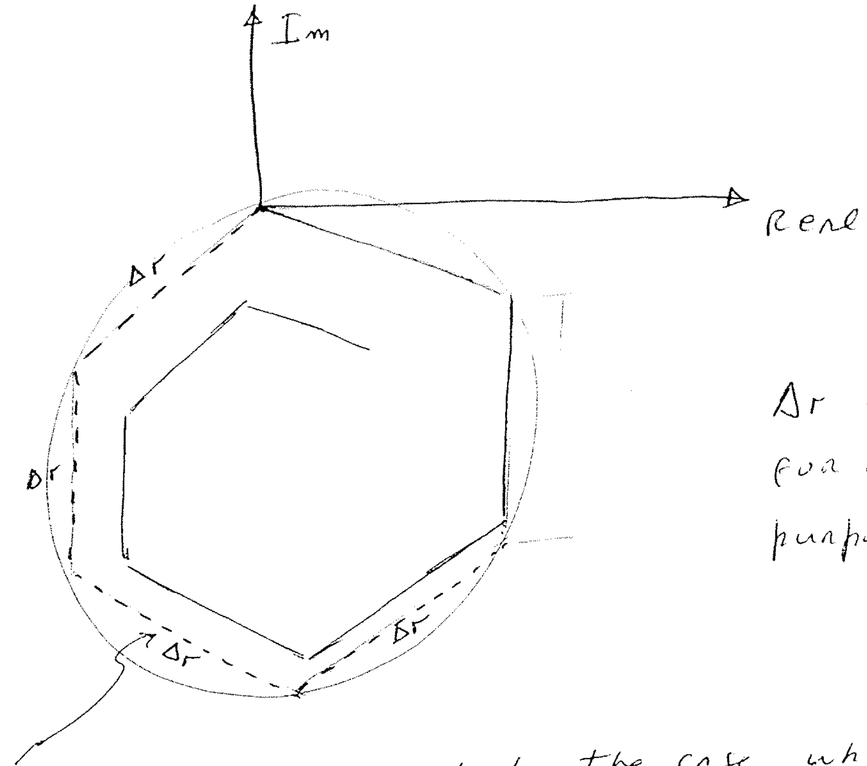
$$\int_{r=z}^{\infty} e^{-i\frac{\omega}{c}r}\,dr = \text{SUM}$$

We realize that adding more and more complex numbers $dr\, e^{-\frac{\omega}{c}\Delta r}$ results in going around the circumference of radius $R = \frac{c}{\omega}$. Thus, we do not get a definite answer!

But $\int_{r=z}^{\infty} e^{-i\frac{\omega}{c}r}\,\eta\,dr$ may have a definite value if we assume tha η (the charge density) taper off as r increases. Let's see why.



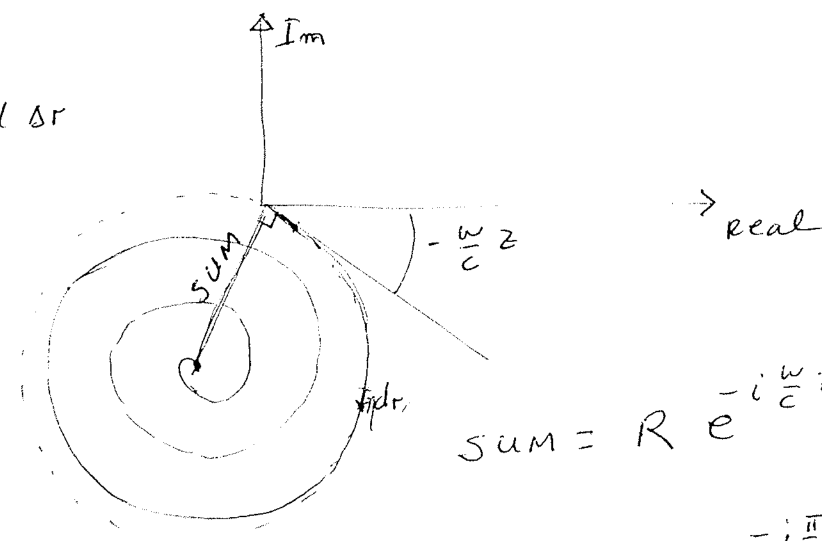




Δr made big for illustrative purpos

The dashed line corresfond to the case where the adding complex number have the same magnitude Δr.
The solid line correspond to the addition of complex numbers whose magnitude decreases progressively.

For small Δr we have:



$$SUM = R\, e^{-i\frac{w}{c}z - i\frac{\pi}{2}}$$

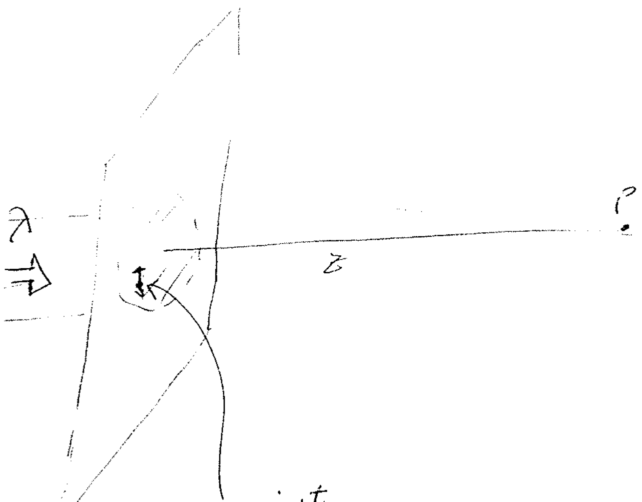
$$= \frac{c}{w} e^{-i\frac{\pi}{2}} e^{-i\frac{w}{c}z}$$

thus

$$\boxed{\int_{r=z}^{\infty} e^{-i\frac{r}{c}\omega}\,\eta\,dr = \eta_0 \frac{c}{\omega}\, e^{-i\frac{\pi}{2}}\, e^{-i\frac{\omega}{c}z} = \eta_0\left(-i\frac{c}{\omega}\right) e^{-i\frac{\omega}{c}z}} \qquad (5)$$

$\eta_0 = \eta(r=z)$

Using (5) in expression (4)



$$\boxed{E(\text{at } P) = -\frac{q\,\eta_0\,x_0}{2\epsilon_0}\left(i\frac{\omega}{c}\right) e^{i\omega\left(t-\frac{z}{c}\right)}} \qquad (6)$$

$x = x_0 e^{i\omega t}$

$v = i\omega x = i\omega x_0 e^{i\omega t}$

$$\boxed{E(\text{at } P, t) = -\frac{\eta_0 q}{2\epsilon_0 c}\Big[\text{velocity of charges}\Big]_{\text{at } t-\frac{z}{c}}} \qquad (6)'$$