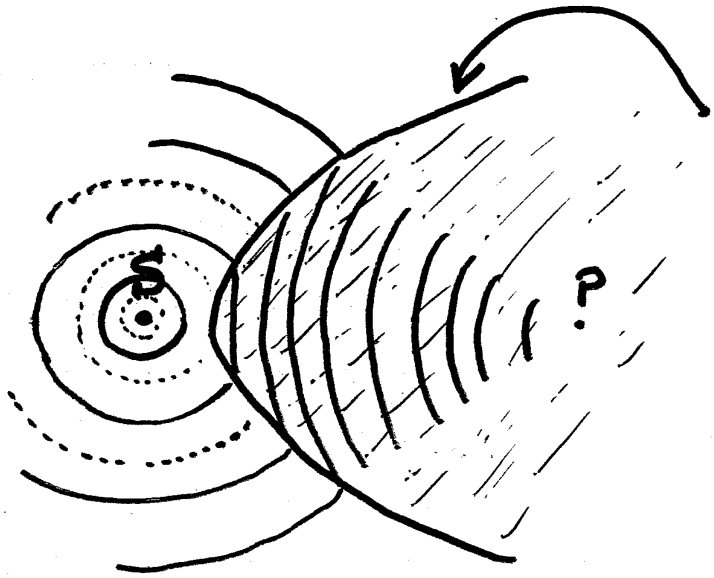
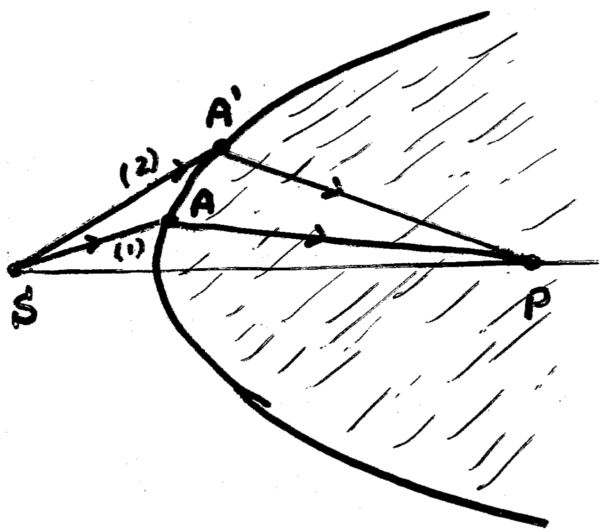
REFRACTION at ASPHERICAL SURFACES 5.2.1



Given S and P, what should be the shape of this surface such that spherical wavefronts outgoing from S become spherical wavefronts converging toward P after refracting at that surface

Shape of the surface, UNKNOWN



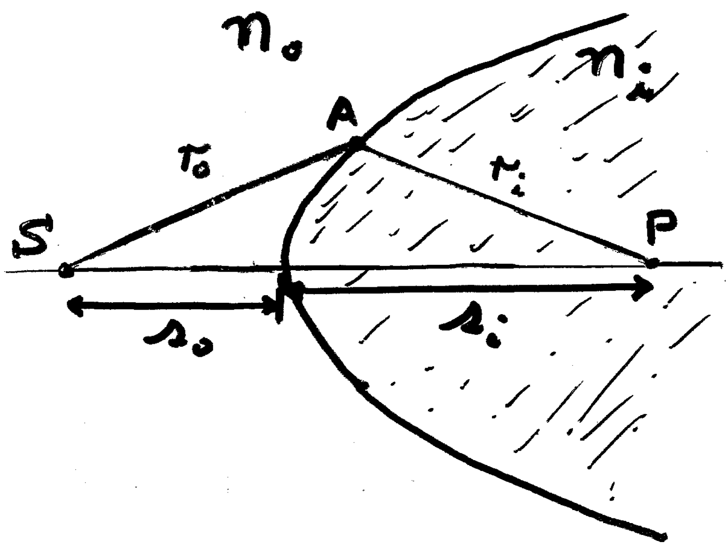
Fermat's Principle indicates that, in going from S to P, the actual path followed by the light ray is the one that makes the optical path (or equivalently, the travel time) stationary with respect to variations or modifications of the actual path.

Since in the graph above we are requiring that <u>All</u> the rays leaving the source S (in particular rays (1) and (2) shown in the figure) reach point P it must occur that, according to the Fermat's Principle, that every single ray SAP, SA'P, ..., etc make the optical path stationary.

That is, it must occur that

OPL (ray-1) = OPL (ray-2) = ...

the, the question becomes:



Given s_o and s_i, what should be the shape of the surface such that the optical path of the ray SAP remains constant regardless of the location of the point A on that surface

$$\text{time of travel} = \frac{r_o}{c/n_o} + \frac{r_i}{c/n_i} = \text{const}$$

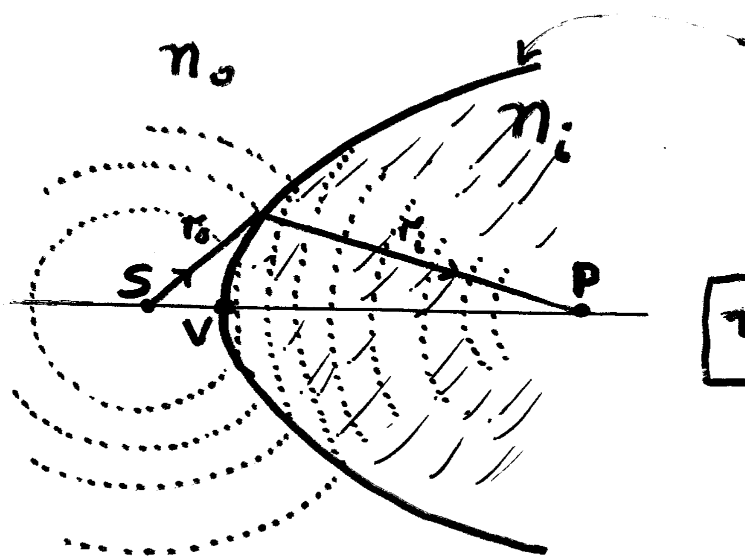
$$= \frac{1}{c}\underbrace{\left(n_o r_o + n_i r_i\right)}_{\text{optical path}} = \text{const}$$

$$\boxed{n_o r_o + n_i r_i = \text{const}}$$

condition to be satisfied by all the rays refracted at the surface

Equation of a Cartesian oval

The significance of Cartesian ovals in Optics was studied by Descartes in the early 1600s



Given S and P (and V) this cartesian oval surface satisfies

$$\boxed{n_o r_o + n_i r_i = \text{const}}$$

Source "S" imaged at "P"

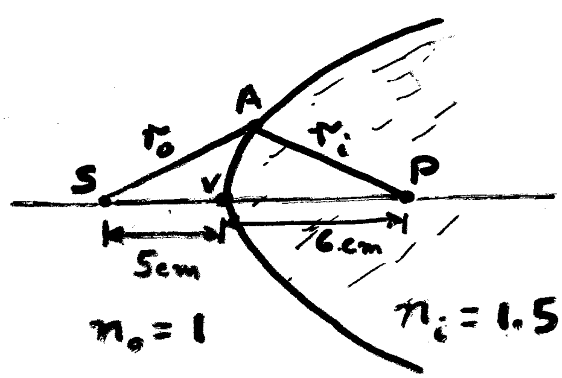
The principle of reversibility (implicit in the Fermat's Principle) implies that a source placed at P would be imaged at S.

Accordingly,

S and P are spoken as conjugate points

Notice: Each couple of points "S" and "P" will generate a different cartesian oval.

<u>Example</u> We wish to construct a cartesian oval such that the conjugate points will be separated by 11 cm when the object is 5 cm from the vertex.
If $n_o = 1$ (air) and $n_1 = 1.5$ (glass) draw several points of the required surface



- Time to go from S to P (through vertice V)

$$\frac{5\,cm}{c/n_o} + \frac{6\,cm}{c/n_1} =$$

$$= \frac{5cm + 1.5 \times 6\,cm}{c} = t\,(SVP)$$

- Time to go from S to P through A

$$t\,(SAP) = \frac{r_o}{c/n_o} + \frac{r_i}{c/n_i} = \frac{r_o + 1.5\,r_i}{c} = \frac{\text{optical path length}}{c}$$

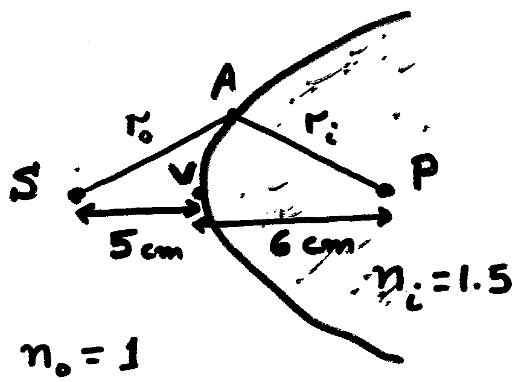
- We require both rays to reach P at the same time

$$t\,(SVP) = t\,(SAP)$$

$\Longleftrightarrow$
equality of optical path length

$$(1)5\,cm + (1.5)\,6\,cm = (1)\,r_o + (1.5)\,r_i$$

$$14\ cm = r_o + 1.5\,r_i$$

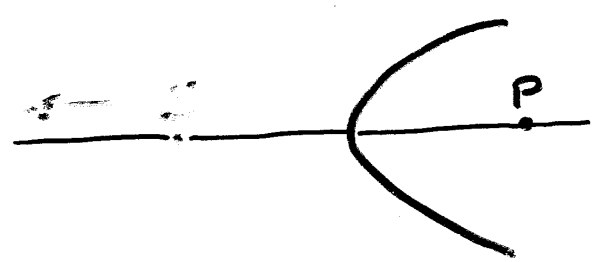


$$\boxed{14\text{cm} = r_o + 1.5\, r_i}$$

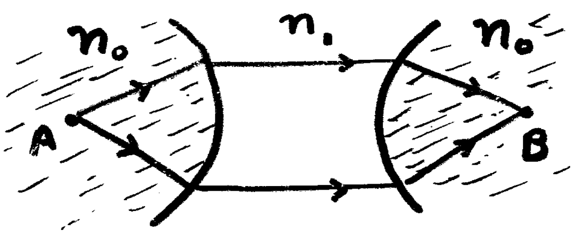
This condition will determine the position of the points A that will generate the oval.

r_o	$r_i = \frac{3}{2}(14\text{cm} - r_o)$	
5cm	6 cm	point V
6	16/3	
7	14/3	
8	12/3	
9	10/3	
10		

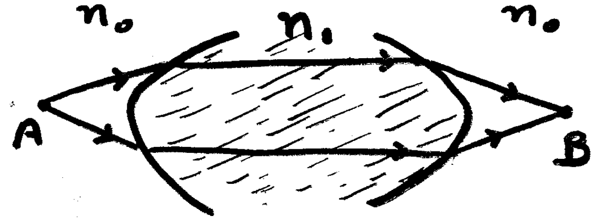
In what follows, our objective is to show that as the source moves to infinity (left side of vertice V) the ovoid surface would gradually metamorphose into an Ellipsoid



the study of those cases (in which the source is at infinity) is important when the applications require that the image be located in the same medium as the object source. We solve such problem by using 2 refracting surface



$n_1 < n_0$

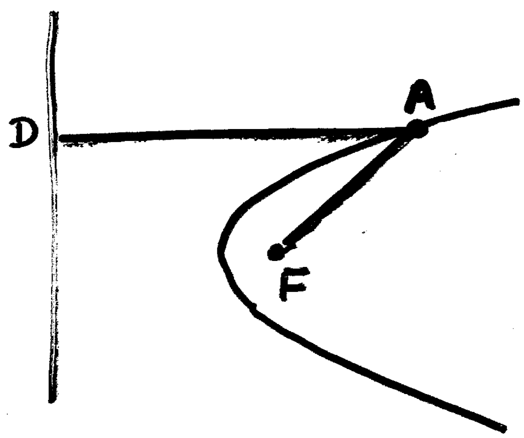


$n_1 > n_0$

So, it is important to study those oval surfaces that refract spherical waves into parallel plane waves

For that matter, let's review some equations related to the ellipse

THE ELLIPSE



Given a plane surface and a point F, determine the locus of points A that satisfy

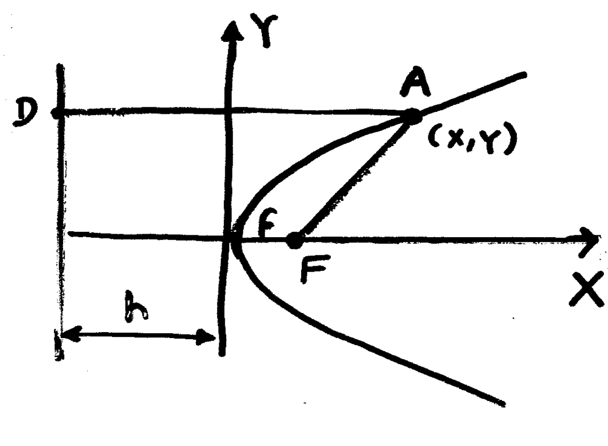
$$\overline{FA} = \epsilon \, \overline{DA}$$

where

ϵ: a given number

$\epsilon < 1$

Considering the symmetry of the problem, let's choose the origen of our system of coordinates as shown in the figure. the origen is the point where the locus intersect the X axis.



$\overline{FA} = \epsilon \overline{DA}$ implies

$$(x-f)^2 + y^2 = \epsilon^2 (x+h)^2$$

$$\underline{x^2} - 2f\underline{\underline{x}} + f^2 + y^2 = \epsilon^2 \underline{x^2} + 2\epsilon^2 h \underline{\underline{x}} + \epsilon^2 h^2$$

$$(1-\epsilon^2)x^2 - 2(f+\epsilon^2 h)x + y^2 = \epsilon^2 h^2 - f^2$$

(when $x=0$, $y=0 \Rightarrow \boxed{f = \epsilon h}$

$$(1-\epsilon^2)x^2 - 2\left(f + \epsilon^2 \frac{f}{\epsilon}\right)x + y^2 = 0$$

$$(1-\epsilon^2)x^2 - 2f(1+\epsilon)x + y^2 = 0$$

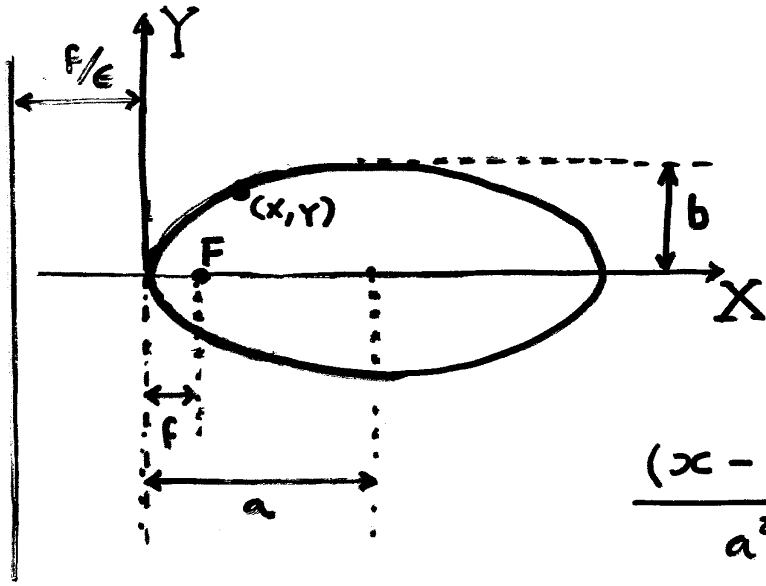
$$x^2 - 2f\frac{1}{1-\epsilon}x + \frac{1}{1-\epsilon^2}y^2 = 0$$

$$\left(x - \frac{f}{1-\epsilon}\right)^2 - \frac{f^2}{(1-\epsilon)^2} + \frac{1}{(1-\epsilon^2)}y^2 = 0$$

$$\left.\frac{\left(x - \frac{f}{1-\epsilon}\right)^2}{\left[\frac{f}{1-\epsilon}\right]^2} + \frac{y^2}{\left[\frac{f}{1-\epsilon}\right]^2(1-\epsilon^2)} = 1 \right\}$$

Ellipse with it center located at

$x_0 = \frac{f}{1-\epsilon}$, $y_0 = 0$



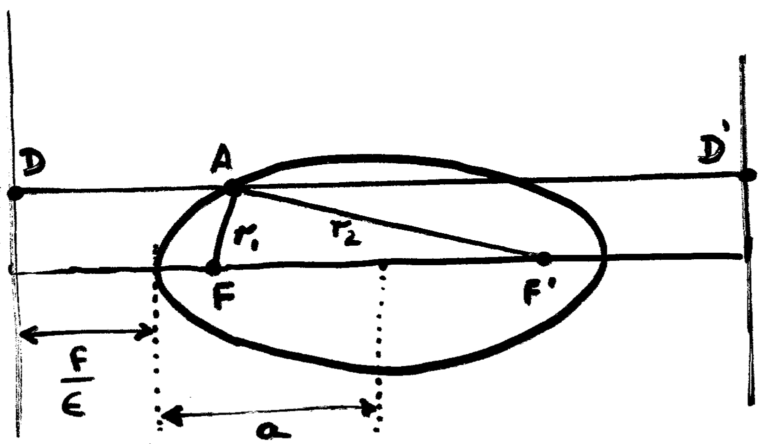
$$b = a\sqrt{1-\epsilon^2}$$

$$\frac{(x-a)^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$a = \frac{f}{1-\epsilon}$$

Notice, by symmetry, that there must exists another point F' such that

$$\overline{F'A} = \epsilon \, \overline{D'A}$$



Now, since $\overline{FA} = \epsilon \, \overline{DA}$, we obtain

$$\overline{FA} + \overline{F'A} = \epsilon\left(\overline{DA} + \overline{D'A}\right)$$

$$= \epsilon \, \overline{DD'} = \text{const}$$

Notice in the figure:

$$\epsilon \, \overline{DD'} = \epsilon \cdot 2 \cdot \left(\frac{f}{\epsilon} + a \right)$$

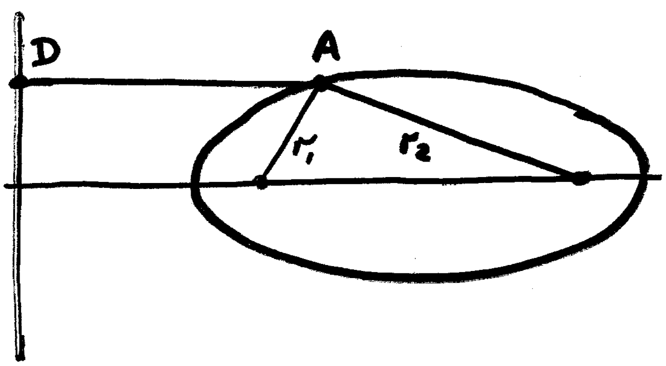
$$= \epsilon \cdot 2 \cdot \left(\frac{f}{\epsilon} + \frac{f}{1-\epsilon} \right) = 2 \underbrace{f \frac{1}{1-\epsilon}}_{a}$$

$$= 2a$$

$$\overline{FA} + \overline{F'A} = 2a$$

$$\boxed{r_1 + r_2 = 2a}$$

The ELLIPSE and its connection with OPTICS



. We know

$$r_1 = \epsilon \overline{DA}$$

where $\epsilon < 1$

. Also

$$r_1 + r_2 = 2a$$

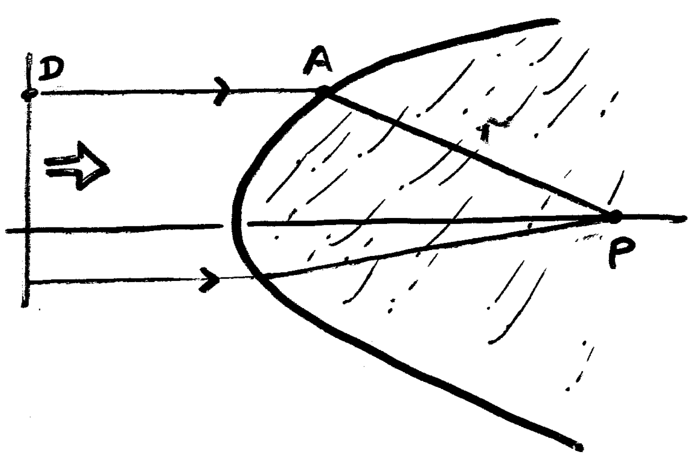
therefore

$$\epsilon \overline{DA} = r_1 = 2a - r_2$$

$$\boxed{\epsilon \overline{DA} + r_2 = 2a = \text{const}} \quad (1)$$

PURE GEOMETRY

Now, suppose we are looking for a surface such that incident plane-waves refract as spherical waves converging toward point P.



According to the Fermat' Principle, our request above implies that all the incident rays must have the same optical path.

that is,

$$n_o \overline{DA} + n_i r = \text{const}$$

or

$$\boxed{\frac{n_o}{n_i} \overline{DA} + r = \text{const}}$$

It has physics content

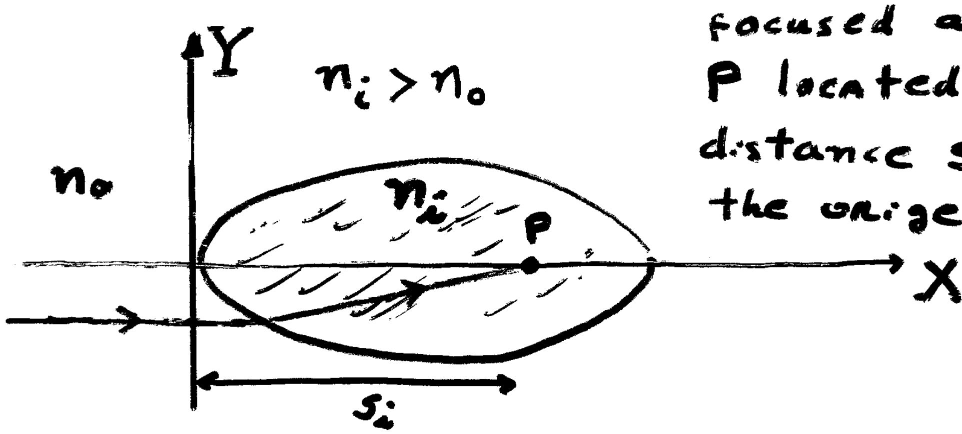
(2)

We identify in the expression (2) the equation of an ellipse (Eq. (1)) provided that

$$\boxed{\frac{n_o}{n_i} < 1}$$

Identifying physical quantities with the eq. of an ellipse

We want incident parallel rays to be focused at point P located at a distance s_i from the origin.



Given n_o, n_i, and s_i find out the parameters of the corresponding ellipsoid refracting surface

Solution:

- $\epsilon = \dfrac{n_o}{n_i}$

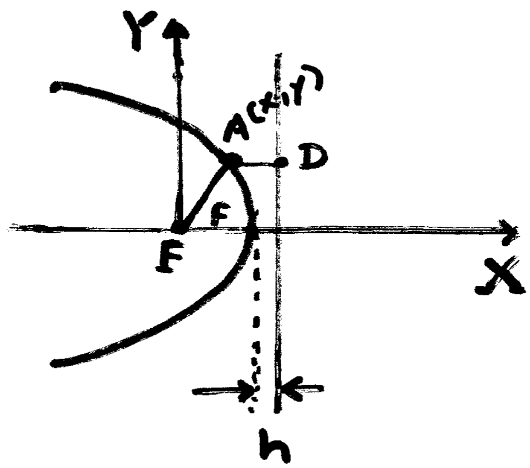
- $s_i \longrightarrow 2a - f = 2\dfrac{f}{1-\epsilon} - f = f\dfrac{1+\epsilon}{1-\epsilon}$

  or

  $$f = \frac{1-\epsilon}{1+\epsilon}\, s_i$$

- the location of the directrix is at $\dfrac{-f}{\epsilon} = \dfrac{-(1-\epsilon)}{1+\epsilon}\dfrac{s_i}{\epsilon}$

THE HYPERBOLA



Find the points (x, y) such that

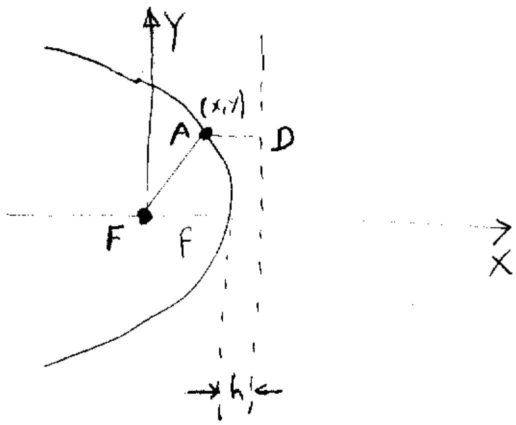
$$\overline{FA} = \epsilon \overline{AD}$$

where $\epsilon > 1$

$$x^2 + y^2 = \epsilon^2 (f + h - x)^2$$

...

About the hyperbola



Find the points (x,y) such that

$$\overline{FA} = e\,\overline{AD} \quad \text{where } \boxed{e>1} \qquad (1)$$

$$x^2+y^2 = e^2\left(f+h-x\right)^2$$

$$x^2+y^2 = e^2(f+h)^2 - 2e^2(f+h)x + e^2x^2$$

$$\Rightarrow \quad y^2 = \left(e^2-1\right)x^2 - 2e^2(f+h)x + e^2(f+h)^2$$

when $x=f$, $y=0 \Rightarrow$ $\quad 0 = (e^2-1)f^2 - 2e^2(f+h)f + e^2(f+h)^2$

too complicated.

It's better to use the graph

$x=f$, $y=0 \Rightarrow$ $\boxed{f = eh}$ ← we verify that this relation satisfies

$$f+h = f + \frac{f}{e} = f\,\frac{e+1}{e}$$

$$y^2 = (e^2-1)x^2 - 2e^2\,f\frac{e+1}{e}\,x + e^2 f^2\frac{(e+1)^2}{e^2}$$

$$= (e^2-1)x^2 - 2ef(e+1)x + f^2(e+1)^2$$

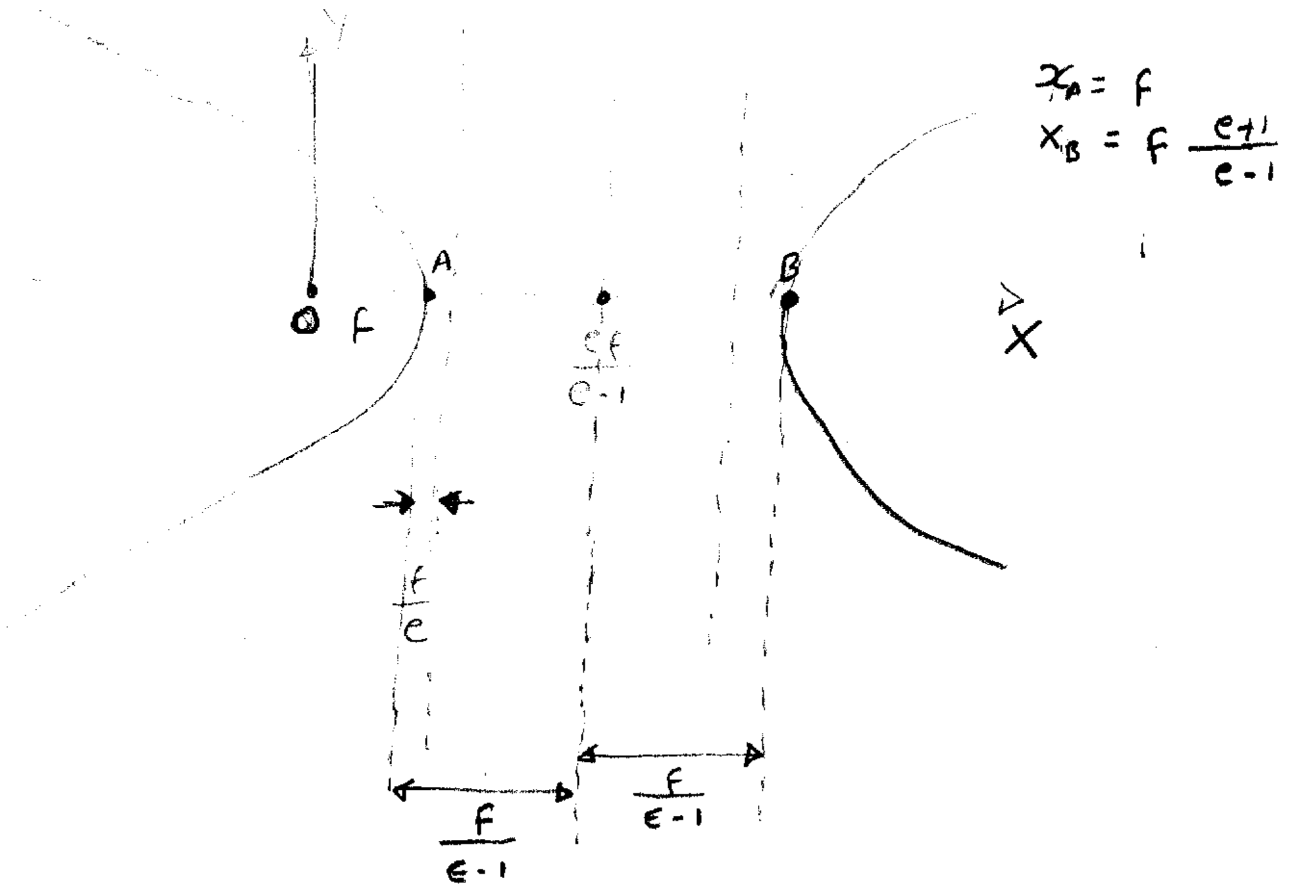
$$\frac{y^2}{e^2-1} = x^2 - 2ef\frac{1}{e-1}x + f^2\frac{e+1}{e-1}$$

$$= \left(x - \frac{ef}{e-1}\right)^2 \underbrace{- \frac{e^2f^2}{(e-1)^2} + f^2\frac{e+1}{e-1}}_{f^2\frac{-e^2+e^2-1}{(e-1)^2} = -\frac{f^2}{(e-1)^2}}$$

$$\frac{y^2}{e^2-1} = \left(x - \frac{ef}{e-1}\right)^2 - \frac{f^2}{(e-1)^2}$$

$$\frac{f^2}{(e-1)^2} = \left(x - \frac{ef}{e-1}\right)^2 - \frac{y^2}{(e^2-1)}$$

$$\Rightarrow \boxed{\frac{\left(x - \frac{ef}{e-1}\right)^2}{f^2/(e-1)^2} - \frac{y^2}{f^2\,(e+1)/e-1} = 1} \qquad (2)$$



when $y=0 \Rightarrow \left(x - \frac{\epsilon f}{\epsilon - 1}\right)^2 = \left(\frac{f}{\epsilon - 1}\right)^2$

$$x = \frac{\epsilon f}{\epsilon - 1} \pm \frac{f}{\epsilon - 1} = \frac{f}{\epsilon - 1}(\epsilon \pm 1)$$

$$x_2 = \frac{f(\epsilon + 1)}{\epsilon - 1} \qquad x_1 = \frac{f}{\epsilon - 1}(\epsilon - 1)$$

$$x_2 = \frac{f(\epsilon + 1)}{\epsilon - 1} \qquad x_1 = f$$

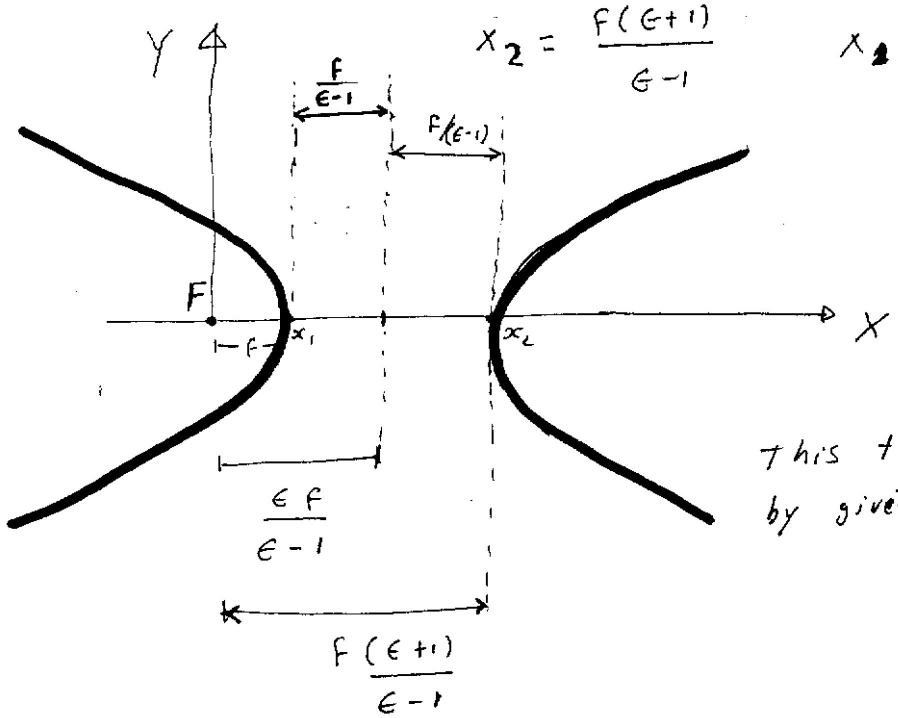
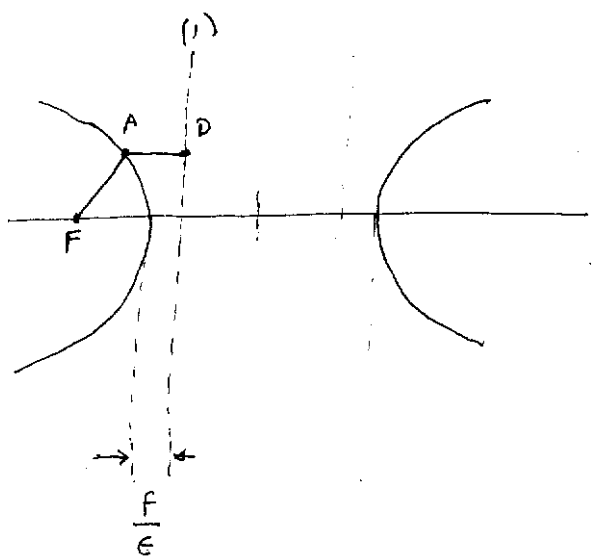


Fig. 1

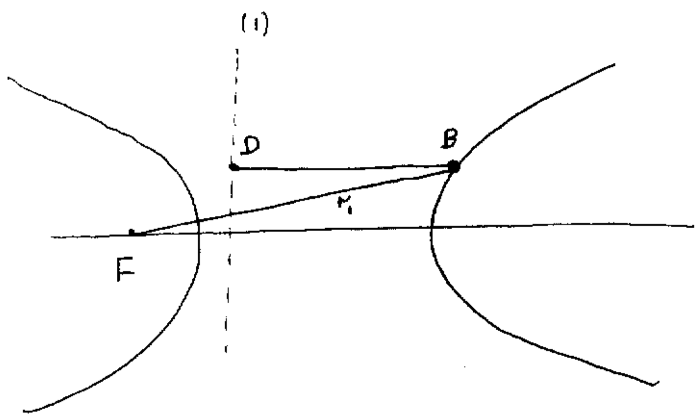
This two curves are generated by giving the values of ϵ and f



Given F and the plane (1), the request that points $A(x,y)$ satisfy

$$\overline{FA} = \epsilon\, \overline{AD} \qquad (3)$$

has generated 2 separated curves



So, it must occur that

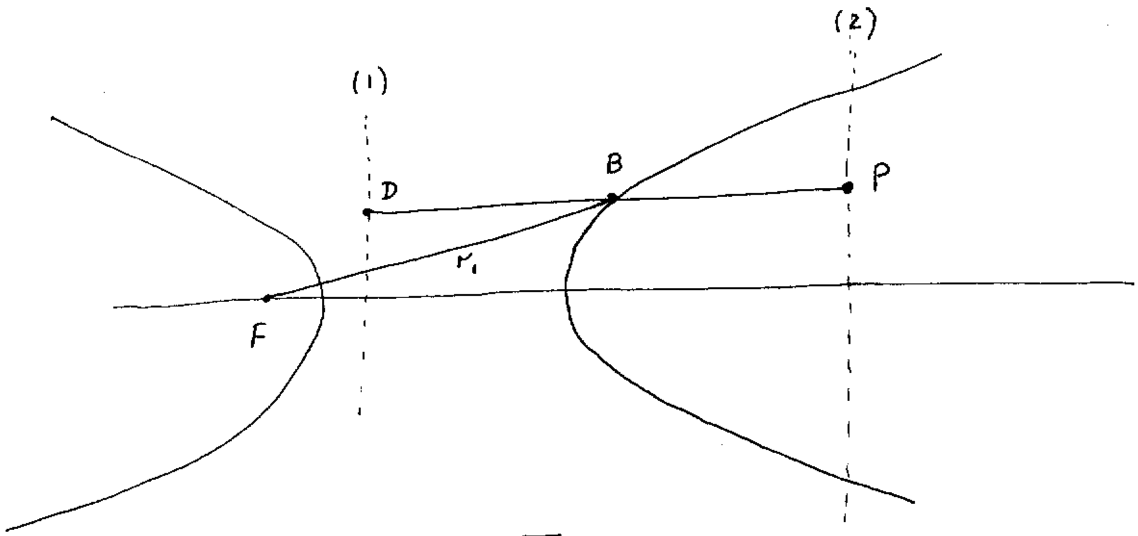
$$\overline{FB} = \epsilon\, \overline{DB} \qquad (4)$$

or

$$r_1 = \epsilon\, \overline{DB}$$

Now it comes the trick that will allow us to make the connection with optics

Lets consider another fixed plane (2) parallel to the directrix (1) as shown in the figure



We know $r_1 = \epsilon\, \overline{DB}$

from the figure we notice

$$= \epsilon\,(\overline{DP} - \overline{BP}) = \epsilon\, \overline{DP} - \epsilon\, \overline{BP}$$

$$r_1 + \epsilon\, \overline{BP} = \epsilon\, \underbrace{\overline{DP}}_{\text{since the plane (2) is fixed, this distance is const}}$$

So

$$\boxed{r_1 + \epsilon\, \overline{BP} = \text{const}} \qquad (5)$$

That is,

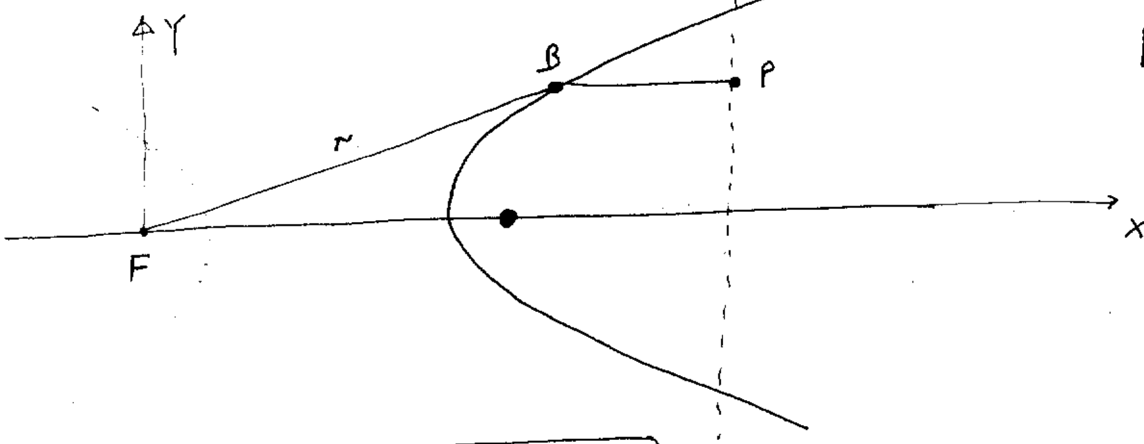


Fig. 2

$$\boxed{r + \epsilon \overline{BP} = \text{const}}$$

This is just pure geometry. (5)

Let's find now the connection with optics.

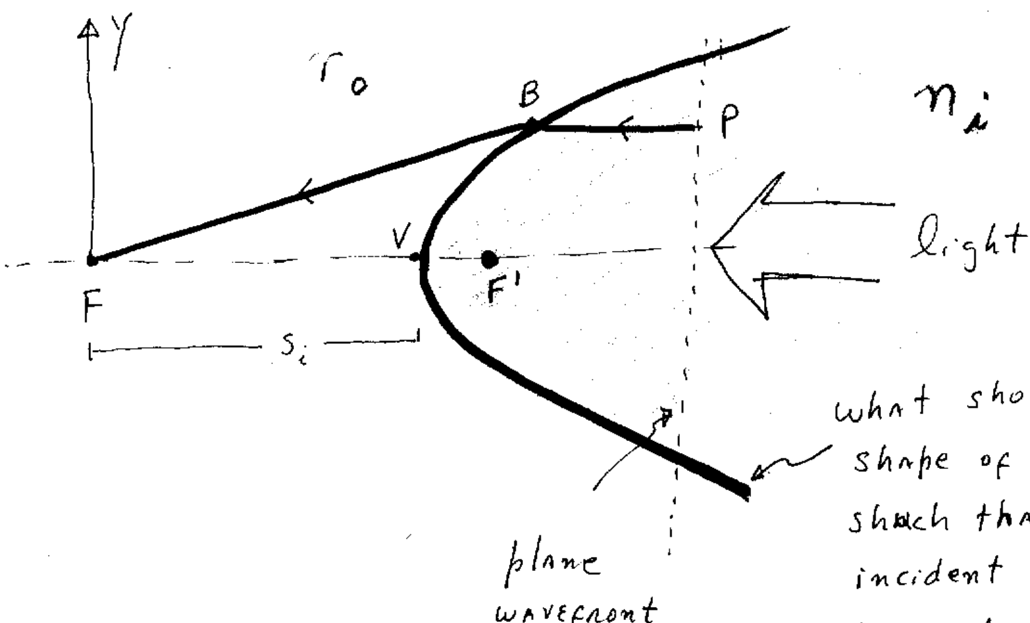


Fig. 3

what should be the shape of this surface such that all the incident beams are focused at point F?

(F is a given point located at a distance s_i from V)

Answer: The points $B(x,y)$ on the surface should satisfy that time travelled by the beam $\overrightarrow{PBF}$ is the same, regardles of the ~~position~~ incident beam $\overrightarrow{PB}$. This is equivalent to say that the OPL (optical path length) for all those rays should be constant.

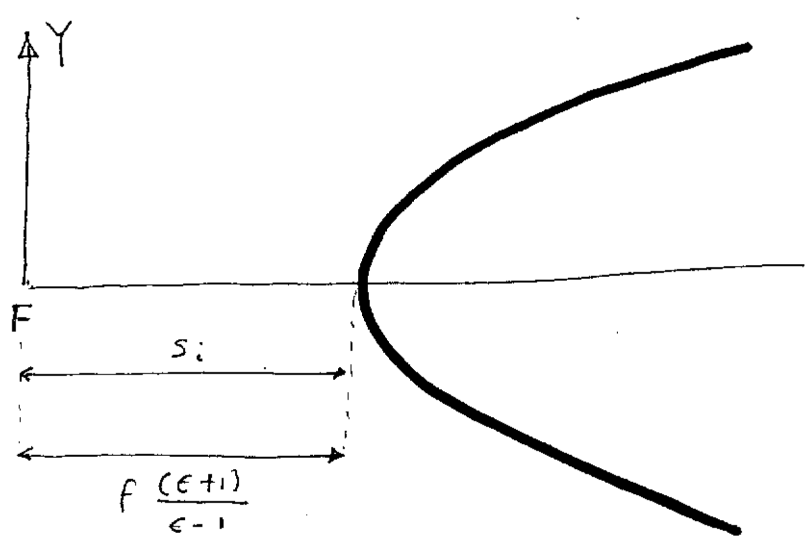
Thus, the condition for the points B is:

$$OPL(\text{of } \overrightarrow{PBF}) = n_i \overrightarrow{PB} + n_0 \overrightarrow{BF} = \text{const}$$

$$\Longleftrightarrow \quad \boxed{\frac{n_i}{n_0} \overline{PB} + \overline{BF} = \text{const}} \qquad (6)$$

If we compare expression (6) with expression (5) we should conclude that the points B specified on expression (6) are on an hyperbolic surface, provided that $\dfrac{n_i}{n_0} > 1$ (that is, we identify $\epsilon = \dfrac{n_i}{n_0}$)

Looking at figures 1 and 3, we identify:



In the assigned problem

$n_i = 1.5$, $n_o = 1$ $\Rightarrow$ $\boxed{\epsilon = \frac{n_i}{n_o} = 1.5}$

We want the light to be focused at $s_i = 10$ cm

$$\Rightarrow \quad 10 \text{ cms} = s_i = f \frac{\epsilon+1}{\epsilon-1}$$

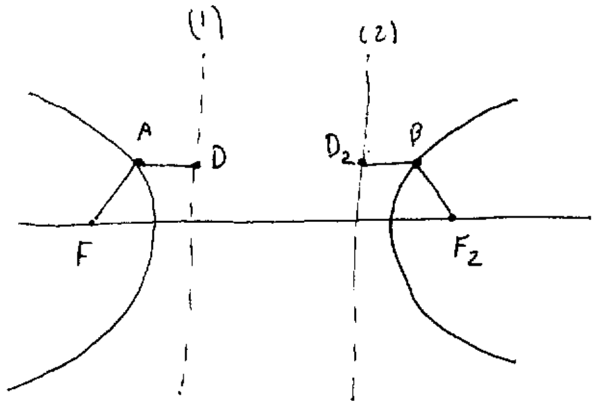
$$= f \frac{1.5+1}{1.5-1} = f \frac{2.5}{0.5} = 5f$$

$$\Rightarrow \quad \boxed{f = 2 \text{ cm}}$$

Given ϵ and f the hyperboloid is determined becaus such surface is given by

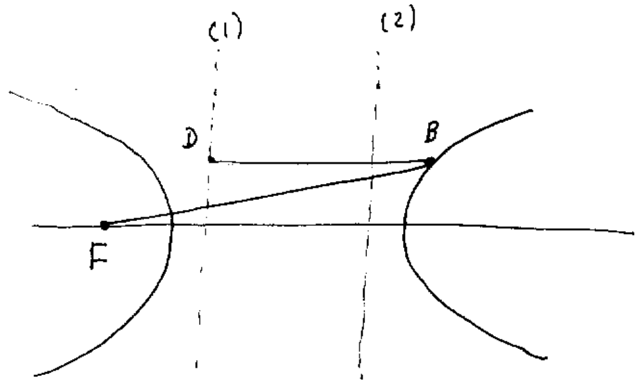
$$\frac{\left(x - \frac{\epsilon f}{\epsilon-1}\right)^2}{\left(f/\epsilon-1\right)^2} - \frac{y^2}{f^2 \frac{\epsilon+1}{\epsilon-1}} = 1$$

SUMMARY A



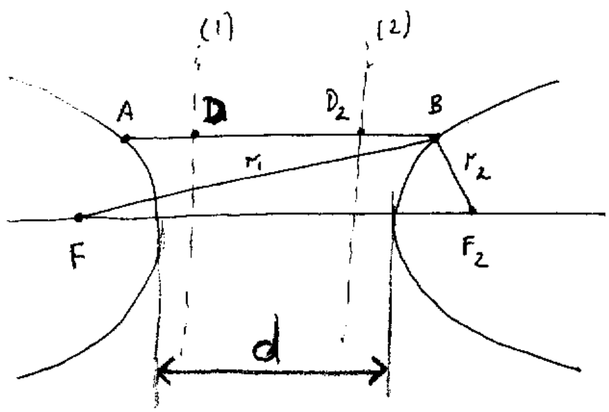
$\overline{F_2 B} = \epsilon \, \overline{BD_2}$ (4)

Notice, either condition (3) or (4) will generate the same couple of curves shown in the figures.
It must be true, then, that



$\overline{FB} = \epsilon \, \overline{DB}$ (because expression (3) generates both curves)

(5)



From (5) and (3)

$r_1 = \epsilon \, \overline{DB}$

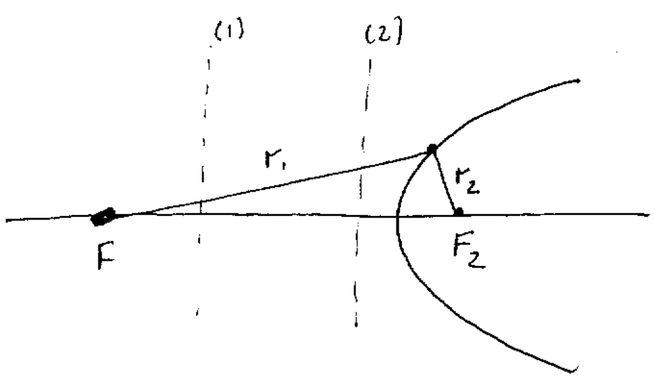
$r_2 = \epsilon \, D_2 B$

$\Rightarrow r_1 - r_2 = \epsilon \, (\overline{DB} - \overline{D_2 B})$

$= \epsilon \, \underbrace{\overline{DD_2}}_{\text{const}}$

that is

$\boxed{r_1 - r_2 = \text{const}}$



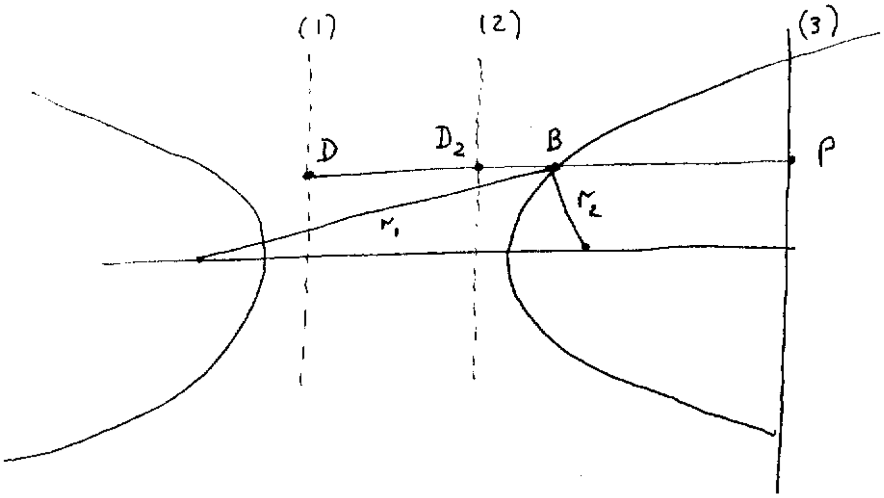
$DD_2 = 2\left(\frac{f}{\epsilon-1} - \frac{f}{\epsilon}\right) = 2f \frac{\epsilon - (\epsilon-1)}{(\epsilon-1)\epsilon}$

$= 2 \frac{f}{(\epsilon-1)\epsilon}$

$\boxed{r_1 - r_2 = 2\frac{f}{\epsilon-1}} = d$

Now it comes the trick that will allow us to make the connection with optics:

Let's consider a plane (3) parallel to the directrices (1) and (2) as shown in the figure below



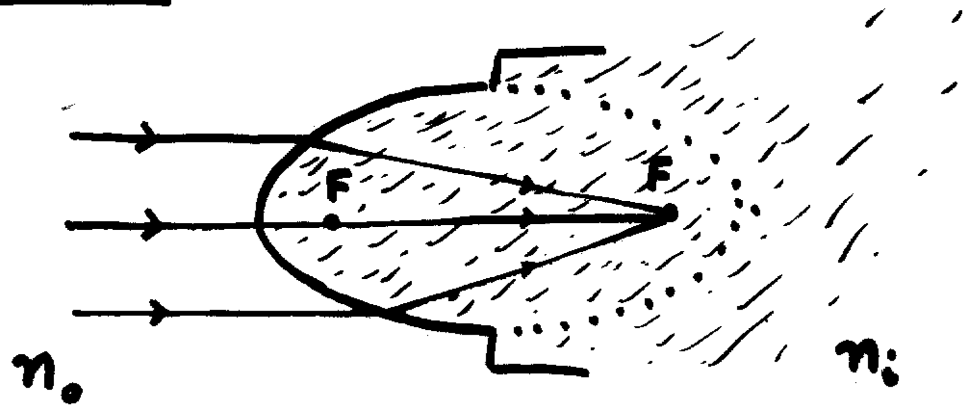
We know
$$r_1 = \epsilon\, \overline{DB}$$
$$= \epsilon\, (\overline{DP} - \overline{BP})$$
$$= \epsilon\, \overline{DP} - \epsilon\, \overline{BP}$$

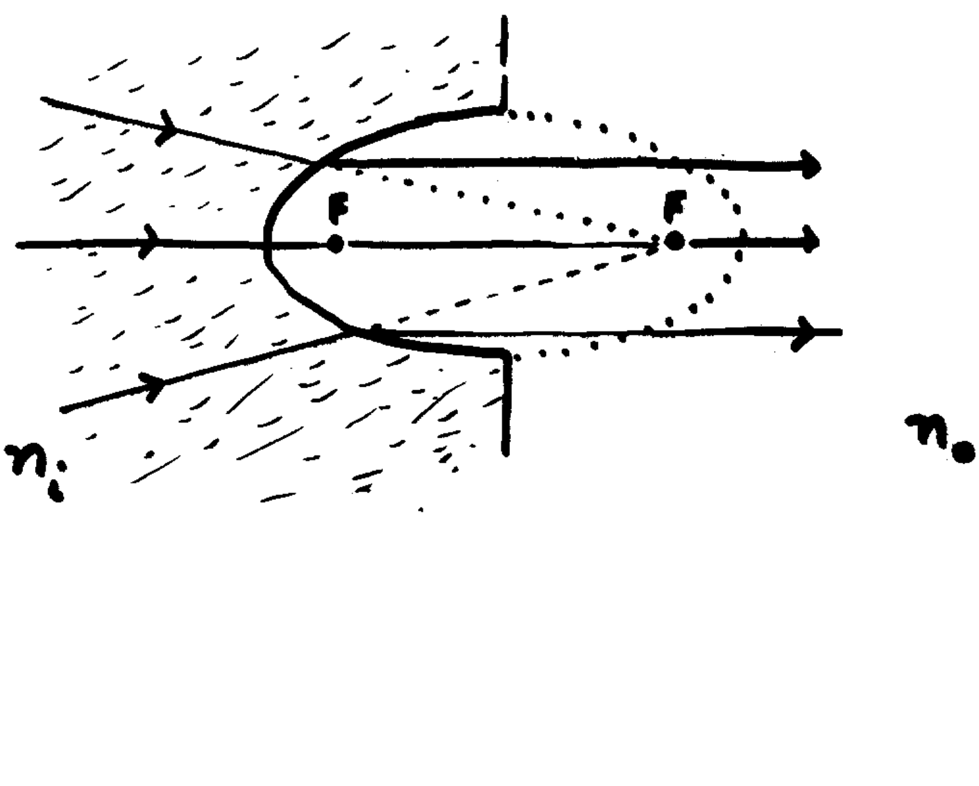
$$r_1 + \epsilon\, \overline{BP} = \underbrace{\epsilon\, \overline{DP}}_{const}$$

$$\boxed{r_1 + \epsilon\, \overline{BP} = const}$$

Ellipsoidal refracting surface

$\boxed{n_0 < n_i}$





Hyperboloidal refracting surface

$\boxed{n_i > n_o}$