

APPLIED OPTICS

Lecture-6 ABERRATIONS

I. Quantifying aberrations

I.A Refraction at a spherical surface

I.B Criteria for evaluating the aberrations

I.C Approximations

I.C.1 Paraxial approximation or Gaussian optics

I.C.2 Third order approximation

I.A Refraction at a spherical surface

Given the distances s_o and s_i , (as shown in the figure below) let's find out let's find out the point "A" that makes the optical path length PAQ stationary.

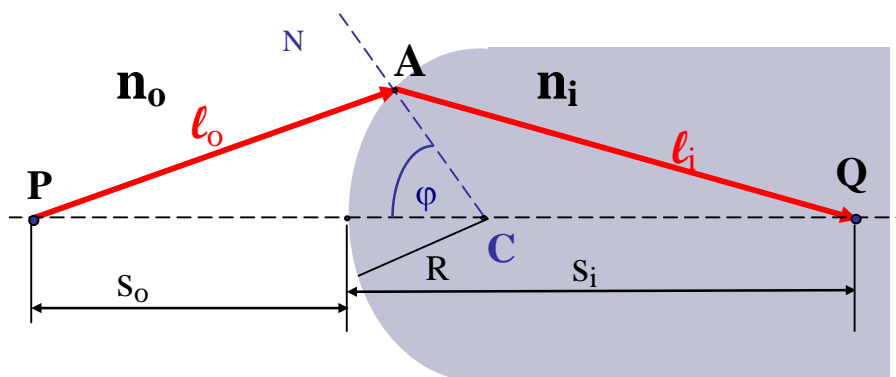


Fig. 1 Refraction through a convex spherical surface of radius R . C is the center. Assume $n_i > n_o$.

$$l = n_o l_o + n_i l_i \quad (1)$$

We want to find the conditions under which l is stationary

The expression for ℓ can be written in terms of R and φ , ℓ_o , and ℓ_i as follows,

$$\begin{aligned}\ell_o^2 &= |\vec{PA}|^2 = |\vec{PC} + \vec{CA}|^2 = |\vec{PC}|^2 + |\vec{CA}|^2 + 2\vec{PC} \cdot \vec{CA} \\ &= (s_o + R)^2 + R^2 - 2(s_o + R)R \cos(\varphi) \quad \text{exact formula} \quad (2)\end{aligned}$$

$$\begin{aligned}\ell_i^2 &= |\vec{AQ}|^2 = |\vec{AC} + \vec{CQ}|^2 = |\vec{AC}|^2 + |\vec{CQ}|^2 + 2\vec{AC} \cdot \vec{CQ} \\ &= R^2 + (s_i - R)^2 + 2R(s_i - R) \cos(\varphi) \quad \text{exact formula} \quad (3)\end{aligned}$$

Hence,

$$\begin{aligned}\ell &= n_o \left[(s_o + R)^2 + R^2 - 2(s_o + R)R \cos(\varphi) \right] + \\ &\quad n_i \left[(s_i - R)^2 + R^2 + 2(s_i - R)R \cos(\varphi) \right] \quad (4)\end{aligned}$$

The formula above is exact; there are no approximations.

In this case we search the condition of stationary with respect to the angle φ .

$$\frac{d\ell}{d\varphi} = 0$$

This implies,

$$\varphi = 0$$

or

$$\frac{n_o}{\ell_o} + \frac{n_i}{\ell_i} = \frac{1}{R} \left[n_i \frac{s_i}{\ell_i} - n_o \frac{s_o}{\ell_o} \right] \quad \text{Lens Eq. (exact formula)} \quad (5)$$

Relationship to be satisfied by the ray PAQ that, upon refraction on a spherical surface, makes the optical path length $\ell = n_o \ell_o + n_i \ell_i$ stationary.

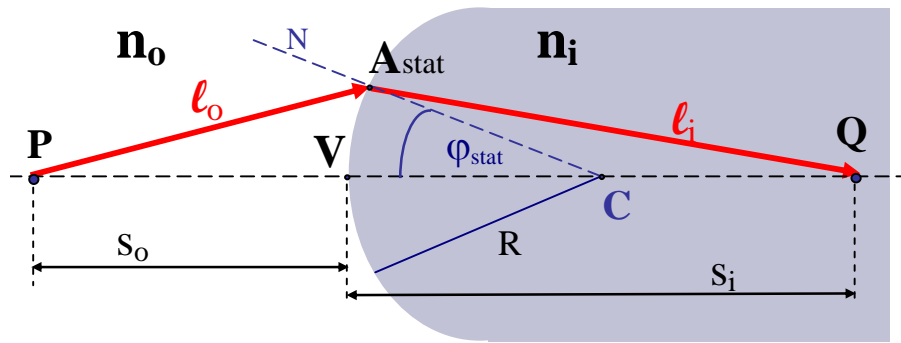


Fig. 2 Given s_o and s_i , the angle ϕ that makes the OPL stationary is either $\phi = 0$ or $\phi = \phi_{\text{stat}}$ (the latter implicitly given in expression (5) above).

In short, we have found that.

for a given spherical refracting surface and given two points P and Q,
 there exist two rays (PVQ and $PA_{\text{stat}}Q$, shown in Fig 2 above) that make their corresponding optical path length stationary.

That is, rays from P that do not deviate too much from either stationary paths PVQ or $PA_{\text{stat}}Q$, will arrive very close to Q (see Fig.3 below). For any other arbitrary point X (different than V or A_{stat}) on the surface, this will not happen. Thus, in general, a spherical refracting surface will produce aberrations (that is, not all the hypothetical rays PXQ, with X at the spherical surface, have the same traveling time.)

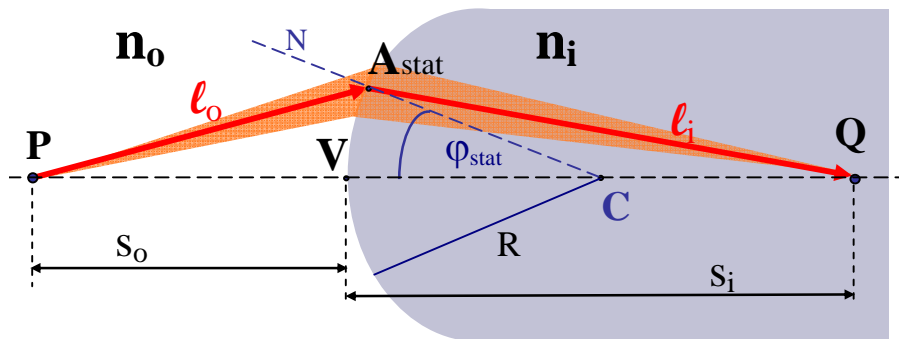


Fig. 3 Rays closer to the ray $PA_{\text{stat}}Q$ travel from P to Q in about the same time. Similarly for the rays around PVQ.

I.B Criteria for evaluating the aberrations

When using a spherical surface instead of an oval, the image of P at Q will be blurred (unless we restrict ourselves to the use of only those rays very close to either PVQ or PA_{stat}Q, as described in the section above.)

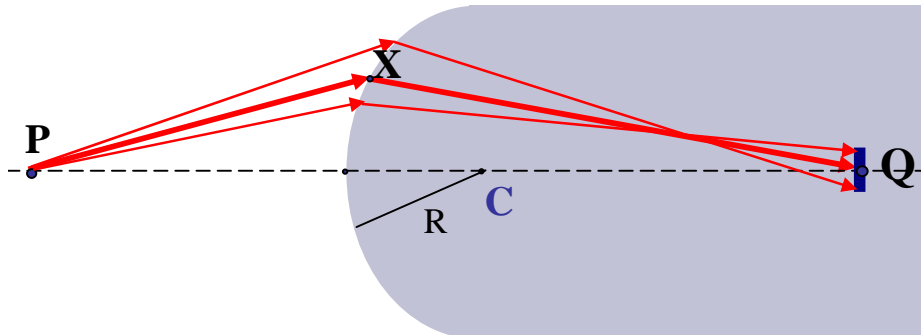


Fig. 4 Rays leaving from P and striking the sphere will produce a blurred image.

An estimation of the degree of aberration we are introducing (for using a **spherical** surface instead of an **oval**) can be given by the difference in the optical path lengths associated to the refractions from these two surfaces; as explained below:

Imagine we were obtaining a perfect image of P at Q by using an oval (see Fig. 5 below.) Let's take an arbitrary point S on the oval.

Now we gradually deform the oval until it adopts a spherical shape of radius R.

If we extrapolate the segment QS, it will intersect the sphere at X.

Arbitrarily, we quantify the aberration as follows,

$$\Delta_{\text{OPL}} = \text{OPL}(\text{PXQ}) - \text{OPL}(\text{PSQ}) \quad (6)$$

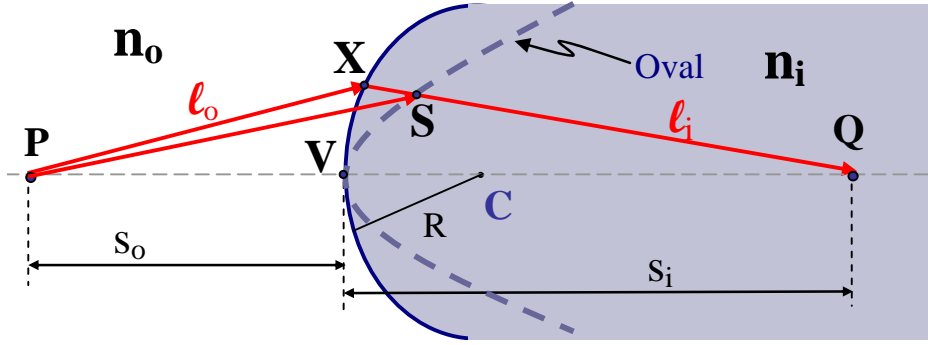


Fig. 5 Comparison between an oval and a spherical surfaces for evaluating the degree of image aberration caused by the ray \vec{PX} .

Since the $OP\mathcal{L}(PSQ)$ refers to the oval, it can be calculated using any point on the oval, in particular point V located in the optical axis. Thus, $OP\mathcal{L}(PSQ) = OP\mathcal{L}(PVQ) = n_0 s_o + n_i s_i$. Accordingly,

$$\begin{aligned} \Delta_{OPL} &= OP\mathcal{L}(\vec{PXQ}) - OP\mathcal{L}(\vec{PSQ}) \\ &= (n_0 \ell_o + n_i \ell_i) - (n_0 s_o + n_i s_i) \\ &= n_0 (\ell_o - s_o) + n_i (\ell_i - s_i) \end{aligned} \quad (7)$$

The expression for Δ_{OPL} given above is a way to quantify the aberration produced by the ray \vec{PX} (with respect to the ray \vec{PS}); see Fig 5.)

I.C Approximations

It is a bit cumbersome to work with the (exact) lens equation $\frac{n_o}{\ell_o} + \frac{n_i}{\ell_i} = \frac{1}{R} \left[n_i \frac{s_i}{\ell_i} - n_o \frac{s_o}{\ell_o} \right]$ (Eq. 5) when using the (exact) formulas for ℓ_o , and ℓ_i , (Eqs. 2 and 3,)

$$\ell_o^2 = (s_o + R)^2 + R^2 - 2 (s_o + R) R \cos(\varphi)$$

$$\ell_i^2 = R^2 + (s_i - R)^2 + 2 R(s_i - R) \cos(\varphi)$$

Simpler expressions result when considering particular cases in which the point X is closer to the optical axis. How much close? Let's say a distance "h" that is small compared to R, s_o and s_i.

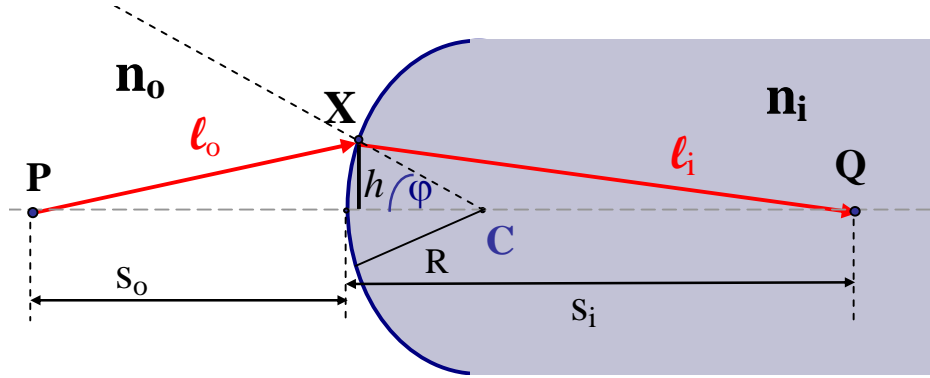


Fig. 6 Rays that refract at points close to the optical axis.

For small h/R, $\text{Cos}(\varphi) = [1 - \text{Sin}^2\varphi]^{1/2} = [1 - (h/R)^2]^{1/2}$

$$\text{Cos}(\varphi) = 1 - \frac{1}{2} \left(\frac{h}{R}\right)^2 - \frac{1}{8} \left(\frac{h}{R}\right)^4 + \dots \quad (8)$$

Replacing (8) in Eqs. 2 and 3 one obtains (see Appendix-1),

$$\ell_o = s_o \left[1 + \left(\frac{1}{R} + \frac{1}{s_o}\right) \frac{R^2}{s_o} \left(\frac{h}{R}\right)^2 + \frac{1}{4} \left(\frac{1}{R} + \frac{1}{s_o}\right) \frac{R^2}{s_o} \left(\frac{h}{R}\right)^4 + \dots \right]^{1/2}$$

and (9)

$$\ell_i = s_i \left[1 - \left(\frac{1}{R} - \frac{1}{s_i}\right) \frac{R^2}{s_i} \left(\frac{h}{R}\right)^2 - \frac{1}{4} \left(\frac{1}{R} - \frac{1}{s_i}\right) \frac{R^2}{s_i} \left(\frac{h}{R}\right)^4 + \dots \right]^{1/2}$$

Under the approximation (8), let's find also a corresponding expression for the degree of aberration (Eq. 7), $\Delta_{\text{OPL}} = n_o (\ell_o - s_o) + n_i (\ell_i - s_i)$.

First, let's proceed with an additional approximation,

$$\ell_o = s_o \left[1 + \left(\frac{1}{R} + \frac{1}{s_o}\right) \frac{R^2}{s_o} \left(\frac{h}{R}\right)^2 + \frac{1}{4} \left(\frac{1}{R} + \frac{1}{s_o}\right) \frac{R^2}{s_o} \left(\frac{h}{R}\right)^4 + \dots \right]^{1/2}$$

using $(1+x)^{1/2} = 1 + (1/2)x - (1/8)x^2 + \dots$,

$$\ell_o = s_o \left[1 + \frac{1}{2} \left(\left(\frac{1}{R} + \frac{1}{s_o} \right) \frac{R^2}{s_o} \left(\frac{h}{R} \right)^2 + \frac{1}{4} \left(\frac{1}{R} + \frac{1}{s_o} \right) \frac{R^2}{s_o} \left(\frac{h}{R} \right)^4 \right) + \right. \\ \left. - \frac{1}{8} \left(\left(\frac{1}{R} + \frac{1}{s_o} \right)^2 \left(\frac{R^2}{s_o} \right)^2 \left(\frac{h}{R} \right)^4 \right) + \dots + \mathcal{O} \left(\frac{h}{R} \right)^6 \right] \quad (10)$$

Similarly,

$$\ell_i = s_i \left[1 - \left(\frac{1}{R} - \frac{1}{s_i} \right) \frac{R^2}{s_i} \left(\frac{h}{R} \right)^2 - \frac{1}{4} \left(\frac{1}{R} - \frac{1}{s_i} \right) \frac{R^2}{s_i} \left(\frac{h}{R} \right)^4 + \dots \right]^{1/2}$$

using $(1+x)^{1/2} = 1 + (1/2)x - (1/8)x^2 + \dots$,

$$\ell_i = s_i \left[1 - \frac{1}{2} \left(\left(\frac{1}{R} - \frac{1}{s_i} \right) \frac{R^2}{s_i} \left(\frac{h}{R} \right)^2 - \frac{1}{4} \left(\frac{1}{R} - \frac{1}{s_i} \right) \frac{R^2}{s_i} \left(\frac{h}{R} \right)^4 \right) + \right. \\ \left. - \frac{1}{8} \left(\left(\frac{1}{R} - \frac{1}{s_i} \right)^2 \left(\frac{R^2}{s_i} \right)^2 \left(\frac{h}{R} \right)^4 \right) + \dots + \mathcal{O} \left(\frac{h}{R} \right)^6 \right] \quad (11)$$

Hence,

$$\Delta_{\text{OPL}} = n_o (\ell_o - s_o) + n_i (\ell_i - s_i) \\ = \frac{R^2}{2} \left[\left(\frac{n_o - n_i}{R} + \frac{n_o}{s_o} + \frac{n_i}{s_i} \right) \right] \left(\frac{h}{R} \right)^2 + \frac{R^2}{8} \left[\frac{n_o - n_i}{R} + \frac{n_o}{s_o} + \frac{n_i}{s_i} \right] \left(\frac{h}{R} \right)^4 \\ - \frac{R^4}{8} \left[\left(\frac{1}{R} + \frac{1}{s_o} \right)^2 \frac{n_o}{s_o} + \left(\frac{1}{R} - \frac{1}{s_i} \right)^2 \frac{n_i}{s_i} \right] \left(\frac{h}{R} \right)^4 + \mathcal{O} \left(\frac{h}{R} \right)^6 \quad (12)$$

1.C.1 Paraxial approximation or Gaussian optics

In this approximation the quadratic terms $(h/r)^2$ are put aside. Thus,

$$\text{Cos}(\varphi) = 1 \quad \text{and} \quad \text{Sin}(\varphi) = \varphi \quad (\varphi \text{ in radians.}) \quad (13)$$

Paraxial approximation

Under this approximation, Eqs (11) and (12) indicate,

$$\ell_o = s_o \quad \text{and} \quad \ell_i = s_i$$

and the lens equation (5) becomes

$$\frac{n_o}{s_o} + \frac{n_i}{s_i} = \frac{n_i - n_o}{R} \quad \text{Lens equation} \quad (14)$$

(in the paraxial approximation)

Notice that “h”, the distance from X to the optical axis, is not involved in (14). This means that all the rays from P that hit the surface (at points that satisfy the paraxial approximation (13)) will be “perfectly” imaged at point Q.

Optics has gone a long way even with this approximation. It is what constitutes the Gaussian Optics.

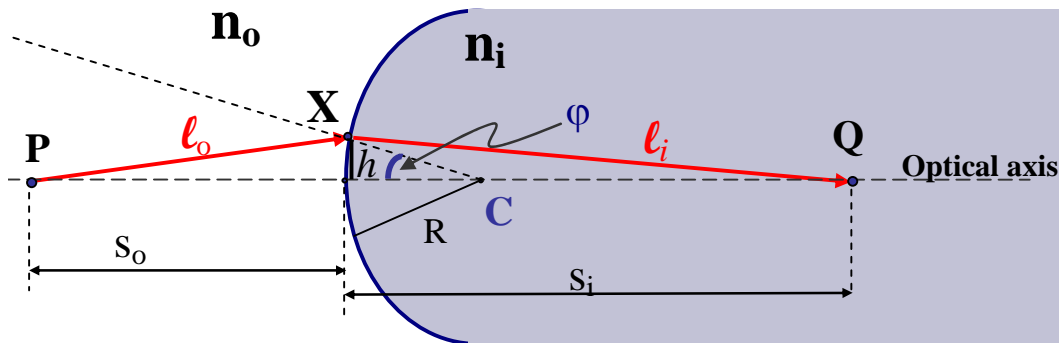


Fig. 7 Paraxial approximation. Points X such that $\text{Sin } \phi \sim \phi$ (i.e. $(h/R) \ll 1$) are “perfectly” imaged at Q.

The graph below shows that for small values of ϕ , the error of assuming that $\text{Sin } \phi = \phi$ incurs in an error of less than $\sim 0.02/0.32 = 6\%$ if we keep the angle ϕ smaller than 20 degrees (which is not such a small angle after all.)

When pressed to provide a more accurate estimation of the aberration under the paraxial approximation, we can use expression (12). Notice that under the condition (14) the error in the OPL given by (12) becomes

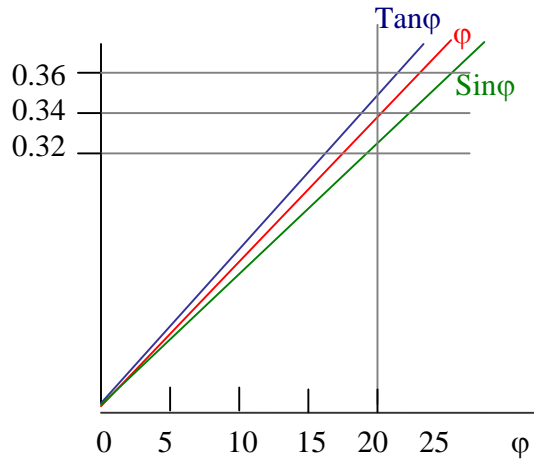


Fig. 8 Paraxial approximation. For $\varphi \sim 20^\circ$ or less, the approximation of $\text{Sin } \varphi \sim \varphi$ is made with an error of $\sim 6\%$ or less.

$$\begin{aligned} \Delta_{\text{OPL}} &= n_o (\ell_o - s_o) + n_i (\ell_i - s_i) \\ &= -\frac{R^4}{8} \left[\left(\frac{1}{R} + \frac{1}{s_o} \right)^2 \frac{n_o}{s_o} + \left(\frac{1}{R} - \frac{1}{s_i} \right)^2 \frac{n_i}{s_i} \right] \left(\frac{h}{R} \right)^4 + \mathcal{O} \left(\frac{h}{R} \right)^6 \end{aligned} \quad (15)$$

That is, for the aberration to be significant, not h/R , or $(h/R)^2$, neither $(h/R)^3$ have to be significant. It is the term $(h/R)^4$ what has to be significant. No wonder why the formula $\frac{n_o}{\ell_o} + \frac{n_i}{\ell_i} = \frac{n_i - n_o}{R}$ has gone

a long way in Optics.

I.C.2 Third order approximation

$$\begin{aligned} \text{Cos}(\varphi) &= 1 - \frac{1}{2} \varphi^2 \\ \text{Sin}(\varphi) &= \varphi - \frac{1}{3} \varphi^3 \end{aligned} \quad (16)$$

From (10),

$$\frac{1}{\ell_o} = \frac{1}{s_o} \left[1 + \left(\frac{1}{R} + \frac{1}{s_o} \right) \frac{R^2}{s_o} \left(\frac{h}{R} \right)^2 + \frac{1}{4} \left(\frac{1}{R} + \frac{1}{s_o} \right) \frac{R^2}{s_o} \left(\frac{h}{R} \right)^4 + \dots \right]^{-1/2}$$

keeping only the quadratic terms,

$$\frac{1}{\ell_o} = \frac{1}{s_o} \left[1 - \frac{1}{2} \left(\frac{1}{R} + \frac{1}{s_o} \right) \frac{R^2}{s_o} \left(\frac{h}{R} \right)^2 + \mathcal{O} \left(\frac{h}{R} \right)^4 \right] \quad (17)$$

Similarly, from (11)

$$\frac{1}{\ell_i} = \frac{1}{s_i} \left[1 - \left(\frac{1}{R} - \frac{1}{s_i} \right) \frac{R^2}{s_i} \left(\frac{h}{R} \right)^2 - \frac{1}{4} \left(\frac{1}{R} - \frac{1}{s_i} \right) \frac{R^2}{s_i} \left(\frac{h}{R} \right)^4 + \dots \right]^{-1/2}$$

keeping only the quadratic terms,

$$\frac{1}{\ell_i} = \frac{1}{s_i} \left[1 + \frac{1}{2} \left(\frac{1}{R} - \frac{1}{s_i} \right) \frac{R^2}{s_i} \left(\frac{h}{R} \right)^2 \dots \right] \quad (18)$$

The exact lens equation (5) can be written as

$$\begin{aligned} \frac{n_o}{\ell_o} \left(1 + \frac{s_o}{R} \right) + \frac{n_i}{\ell_i} \left(1 - \frac{s_i}{R} \right) &= 0 \\ n_o \left(1 + \frac{s_o}{R} \right) \frac{1}{s_o} \left[1 - \frac{1}{2} \frac{(s_o + R)}{s_o} \frac{R}{s_o} \left(\frac{h}{R} \right)^2 \dots \right] \\ + n_i \left(1 - \frac{s_i}{R} \right) \frac{1}{s_i} \left[1 + \frac{1}{2} \frac{(s_i - R)}{s_i} \frac{R}{s_i} \left(\frac{h}{R} \right)^2 \dots \right] &= 0 \end{aligned}$$

$$\begin{aligned} n_o \left(\frac{1}{s_o} + \frac{1}{R} \right) \left[1 - \frac{1}{2} \left(\frac{1}{R} + \frac{1}{s_o} \right) \frac{R^2}{s_o} \left(\frac{h}{R} \right)^2 \dots \right] \\ + n_i \left(\frac{1}{s_i} - \frac{1}{R} \right) \left[1 + \frac{1}{2} \left(\frac{1}{R} - \frac{1}{s_i} \right) \frac{R^2}{s_i} \left(\frac{h}{R} \right)^2 \dots \right] &= 0 \end{aligned}$$

$$n_o \left(\frac{1}{s_o} + \frac{1}{R} \right) \left[1 - \frac{1}{2} \left(\frac{1}{R} + \frac{1}{s_o} \right) \frac{R^2}{s_o} \left(\frac{h}{R} \right)^2 \dots \right] \\ - n_i \left(\frac{1}{R} - \frac{1}{s_i} \right) \left[1 + \frac{1}{2} \left(\frac{1}{R} - \frac{1}{s_i} \right) \frac{R^2}{s_i} \left(\frac{h}{R} \right)^2 \dots \right] = 0$$

$$n_o \left(\frac{1}{s_o} + \frac{1}{R} \right) - \frac{1}{2} n_o \left(\frac{1}{R} + \frac{1}{s_o} \right)^2 \frac{R^2}{s_o} \left(\frac{h}{R} \right)^2 \\ - n_i \left(\frac{1}{R} - \frac{1}{s_i} \right) - \frac{1}{2} \left(\frac{1}{R} - \frac{1}{s_i} \right)^2 \frac{R^2}{s_i} \left(\frac{h}{R} \right)^2 = 0$$

$$\frac{n_o}{s_o} + \frac{n_i}{s_i} - \frac{n_i - n_o}{R} - \frac{1}{2} n_o \left(\frac{1}{R} + \frac{1}{s_o} \right)^2 \frac{R^2}{s_o} \left(\frac{h}{R} \right)^2 - \frac{1}{2} n_i \left(\frac{1}{R} - \frac{1}{s_i} \right)^2 \frac{R^2}{s_i} \left(\frac{h}{R} \right)^2 = 0$$

$$\frac{n_o}{s_o} + \frac{n_i}{s_i} = \frac{n_i - n_o}{R} + \frac{1}{2} \left[\frac{n_o}{s_o} \left(\frac{1}{R} + \frac{1}{s_o} \right)^2 + \frac{n_i}{s_i} \left(\frac{1}{R} - \frac{1}{s_i} \right)^2 \right] h^2$$

Lens Eq. under the *Third Order Approximation* (19)

APPENDIX-1

$$\ell_o^2 = (s_o + R)^2 + R^2 - 2 (s_o + R) R \cos(\varphi)$$

$$\text{using } \cos(\varphi) = 1 - \frac{1}{2} \left(\frac{h}{R} \right)^2 - \frac{1}{8} \left(\frac{h}{R} \right)^4 + \dots \text{ one obtains}$$

$$\ell_o^2 = (s_o + R)^2 + R^2 - 2 (s_o + R) R \left[1 - \frac{1}{2} \left(\frac{h}{R} \right)^2 - \frac{1}{8} \left(\frac{h}{R} \right)^4 + \dots \right]$$

$$\begin{aligned}
&= (s_o + R)^2 + R^2 - 2 (s_o + R) R \left[1 \right] \\
&\quad - 2 (s_o + R) R \left[-\frac{1}{2} \left(\frac{h}{R} \right)^2 - \frac{1}{8} \left(\frac{h}{R} \right)^4 + \dots \right] \\
&= s_o^2 + 2 (s_o + R) R \left[\frac{1}{2} \left(\frac{h}{R} \right)^2 + \frac{1}{8} \left(\frac{h}{R} \right)^4 + \dots \right]
\end{aligned}$$

Thus

$$\begin{aligned}
\ell_o^2 &= s_o^2 + (s_o + R) R \left(\frac{h}{R} \right)^2 + \frac{1}{4} (s_o + R) R \left(\frac{h}{R} \right)^4 \\
\ell_o &= s_o \left[1 + \frac{(s_o + R) R}{s_o} \left(\frac{h}{R} \right)^2 + \frac{1}{4} \frac{(s_o + R) R}{s_o} \left(\frac{h}{R} \right)^4 + \dots \right]^{1/2} \\
\ell_o &= s_o \left[1 + \left(\frac{1}{R} + \frac{1}{s_o} \right) \frac{R^2}{s_o} \left(\frac{h}{R} \right)^2 + \frac{1}{4} \left(\frac{1}{R} + \frac{1}{s_o} \right) \frac{R^2}{s_o} \left(\frac{h}{R} \right)^4 + \dots \right]^{1/2}
\end{aligned} \tag{A-1}$$

Similarly,

$$\begin{aligned}
\ell_i^2 &= R^2 + (s_i - R)^2 + 2 R(s_i - R) \text{Cos}(\varphi) \\
&\quad \text{using } \text{Cos}(\varphi) = 1 - \frac{1}{2} \left(\frac{h}{R} \right)^2 - \frac{1}{8} \left(\frac{h}{R} \right)^4 + \dots \text{ one obtains} \\
\ell_i^2 &= R^2 + (s_i - R)^2 + 2 R(s_i - R) \left[1 - \frac{1}{2} \left(\frac{h}{R} \right)^2 - \frac{1}{8} \left(\frac{h}{R} \right)^4 + \dots \right] \\
&= R^2 + (s_i - R)^2 + 2 R(s_i - R) \left[1 \right] \\
&\quad + 2 R(s_i - R) \left[-\frac{1}{2} \left(\frac{h}{R} \right)^2 - \frac{1}{8} \left(\frac{h}{R} \right)^4 + \dots \right]
\end{aligned}$$

$$\begin{aligned}
&= s_i^2 - 2 R(s_i - R) \left[\frac{1}{2} \left(\frac{h}{R} \right)^2 + \frac{1}{8} \left(\frac{h}{R} \right)^4 + \dots \right] \\
\ell_i^2 &= s_i^2 - R(s_i - R) \left(\frac{h}{R} \right)^2 - R(s_i - R) \frac{1}{4} \left(\frac{h}{R} \right)^4 + \dots \\
\ell_i &= s_i \left[1 - \frac{(s_i - R) R}{s_i} \left(\frac{h}{R} \right)^2 - \frac{1}{4} \frac{(s_i - R) R}{s_i} \left(\frac{h}{R} \right)^4 + \dots \right]^{1/2} \\
\ell_i &= s_i \left[1 - \left(\frac{1}{R} - \frac{1}{s_i} \right) \frac{R^2}{s_i} \left(\frac{h}{R} \right)^2 - \frac{1}{4} \left(\frac{1}{R} - \frac{1}{s_i} \right) \frac{R^2}{s_i} \left(\frac{h}{R} \right)^4 + \dots \right]^{1/2} \quad (\text{A-2})
\end{aligned}$$

APPENDIX-B

From (A-1)

$$\begin{aligned}
\ell_o &= s_o \left[1 + \frac{(s_o + R) R}{s_o} \left(\frac{h}{R} \right)^2 + \frac{1}{4} \frac{(s_o + R) R}{s_o} \left(\frac{h}{R} \right)^4 + \dots \right]^{1/2} \\
\text{Using } (1+x)^{1/2} &= 1 + (1/2)x - (1/8)x^2 + \dots \\
\ell_o &= s_o \left[1 + \frac{1}{2} \left(\frac{(s_o + R) R}{s_o} \left(\frac{h}{R} \right)^2 + \frac{1}{4} \frac{(s_o + R) R}{s_o} \left(\frac{h}{R} \right)^4 \right) + \right. \\
&\quad \left. - \frac{1}{8} \left(\frac{(s_o + R)^2}{s_o^2} \left(\frac{R}{s_o} \right)^2 \left(\frac{h}{R} \right)^4 \right) + \dots + \mathcal{O} \left(\frac{h}{R} \right)^6 \right] \quad (\text{B-1})
\end{aligned}$$

Similarly,

$$\begin{aligned}
\ell_i &= s_i \left[1 - \frac{(s_i - R) R}{s_i} \left(\frac{h}{R} \right)^2 - \frac{1}{4} \frac{(s_i - R) R}{s_i} \left(\frac{h}{R} \right)^4 + \dots \right]^{1/2} \\
\ell_i &= s_i \left[1 - \frac{1}{2} \left(\frac{(s_i - R) R}{s_i} \left(\frac{h}{R} \right)^2 - \frac{1}{4} \frac{(s_i - R) R}{s_i} \left(\frac{h}{R} \right)^4 \right) + \right. \\
&\quad \left. - \frac{1}{8} \left(\frac{(s_i - R)^2}{s_i^2} \left(\frac{R}{s_i} \right)^2 \left(\frac{h}{R} \right)^4 \right) + \dots + \mathcal{O} \left(\frac{h}{R} \right)^6 \right] \quad (\text{B-2})
\end{aligned}$$

From (B-1) and (B-2),

$$\ell_o - s_o = \left[\frac{1}{2} \frac{(s_o + R)}{s_o} R \right] \left(\frac{h}{R} \right)^2 + \left[+ \frac{1}{8} \frac{(s_o + R)}{s_o} R - \frac{1}{8} \left(\frac{(s_o + R)^2}{s_o^2} s_o \left(\frac{R}{s_o} \right)^2 \right) \right] \left(\frac{h}{R} \right)^4 + \mathcal{O} \left(\frac{h}{R} \right)^6$$

$$\ell_i - s_i = \left[- \frac{1}{2} \frac{(s_i - R)}{s_i} R \right] \left(\frac{h}{R} \right)^2 + \left[- \frac{1}{8} \frac{(s_i - R)}{s_i} R - \frac{1}{8} \left(\frac{(s_i - R)^2}{s_i^2} \left(\frac{R}{s_i} \right)^2 s_i \right) \right] \left(\frac{h}{R} \right)^4 + \mathcal{O} \left(\frac{h}{R} \right)^6$$

$$\ell_o - s_o = \left[\frac{1}{2} \left(\frac{1}{R} + \frac{1}{s_o} \right) R^2 \right] \left(\frac{h}{R} \right)^2 + \left[\frac{1}{8} \left(\frac{1}{R} + \frac{1}{s_o} \right) R^2 - \frac{1}{8} \left(\frac{1}{R} + \frac{1}{s_o} \right)^2 R^2 s_o \left(\frac{R}{s_o} \right)^2 \right] \left(\frac{h}{R} \right)^4 + \mathcal{O} \left(\frac{h}{R} \right)^6$$

$$\ell_i - s_i = \left[- \frac{1}{2} \left(\frac{1}{R} - \frac{1}{s_i} \right) R^2 \right] \left(\frac{h}{R} \right)^2 - \left[\frac{1}{8} \left(\frac{1}{R} - \frac{1}{s_i} \right) R^2 + \frac{1}{8} \left(\frac{1}{R} - \frac{1}{s_i} \right)^2 R^2 \left(\frac{R}{s_i} \right)^2 s_i \right] \left(\frac{h}{R} \right)^4 + \mathcal{O} \left(\frac{h}{R} \right)^6$$

$$\ell_o - s_o = \left[\frac{1}{2} \left(\frac{1}{R} + \frac{1}{s_o} \right) R^2 \right] \left(\frac{h}{R} \right)^2 + \frac{1}{8} \left(\frac{1}{R} + \frac{1}{s_o} \right) R^2 \left[1 - \left(\frac{1}{R} + \frac{1}{s_o} \right) s_o \left(\frac{R}{s_o} \right)^2 \right] \left(\frac{h}{R} \right)^4 + \mathcal{O} \left(\frac{h}{R} \right)^6$$

$$\ell_i - s_i = \left[- \frac{1}{2} \left(\frac{1}{R} - \frac{1}{s_i} \right) R^2 \right] \left(\frac{h}{R} \right)^2 - \frac{1}{8} \left(\frac{1}{R} - \frac{1}{s_i} \right) R^2 \left[1 + \left(\frac{1}{R} - \frac{1}{s_i} \right) \left(\frac{R}{s_i} \right)^2 s_i \right] \left(\frac{h}{R} \right)^4 + \mathcal{O} \left(\frac{h}{R} \right)^6$$

$$\Delta_{\text{OPL}} = n_o (\ell_o - s_o) + n_i (\ell_i - s_i)$$

$$= \left[\frac{1}{2} \left(\frac{n_o - n_i}{R} + \frac{n_o}{s_o} + \frac{n_i}{s_i} \right) R^2 \right] \left(\frac{h}{R} \right)^2 + \left[\frac{1}{8} \left(\frac{n_o - n_i}{R} + \frac{n_o}{s_o} + \frac{n_i}{s_i} \right) R^2 \right] \left(\frac{h}{R} \right)^4$$

$$\begin{aligned}
& -\frac{1}{8}\left(\frac{1}{R} + \frac{1}{s_o}\right) R^2 \left[\left(\frac{1}{R} + \frac{1}{s_o}\right) \left(\frac{n_o}{s_o}\right) R^2 \right] \left(\frac{h}{R}\right)^4 \\
& -\frac{1}{8}\left(\frac{1}{R} - \frac{1}{s_i}\right) R^2 \left[\left(\frac{1}{R} - \frac{1}{s_i}\right) \left(\frac{n_i}{s_i}\right) R^2 \right] \left(\frac{h}{R}\right)^4 + \mathcal{O}\left(\frac{h}{R}\right)^6
\end{aligned}$$

$$\Delta_{\text{OPL}} = n_o (\ell_o - s_o) + n_i (\ell_i - s_i)$$

$$= \frac{R^2}{2} \left[\left(\frac{n_o - n_i}{R} + \frac{n_o}{s_o} + \frac{n_i}{s_i}\right) \right] \left(\frac{h}{R}\right)^2 + \frac{R^2}{8} \left[\frac{n_o - n_i}{R} + \frac{n_o}{s_o} + \frac{n_i}{s_i} \right] \left(\frac{h}{R}\right)^4$$

$$-\frac{R^4}{8} \left[\left(\frac{1}{R} + \frac{1}{s_o}\right)^2 \frac{n_o}{s_o} + \left(\frac{1}{R} - \frac{1}{s_i}\right)^2 \frac{n_i}{s_i} \right] \left(\frac{h}{R}\right)^4 + \mathcal{O}\left(\frac{h}{R}\right)^6 \quad (\text{B-3})$$