MAXWELL's EQUATIONS

ME in INTEGRAL FORM

Line integral

Surface integral

ME in DIFFERENTIAL FORM

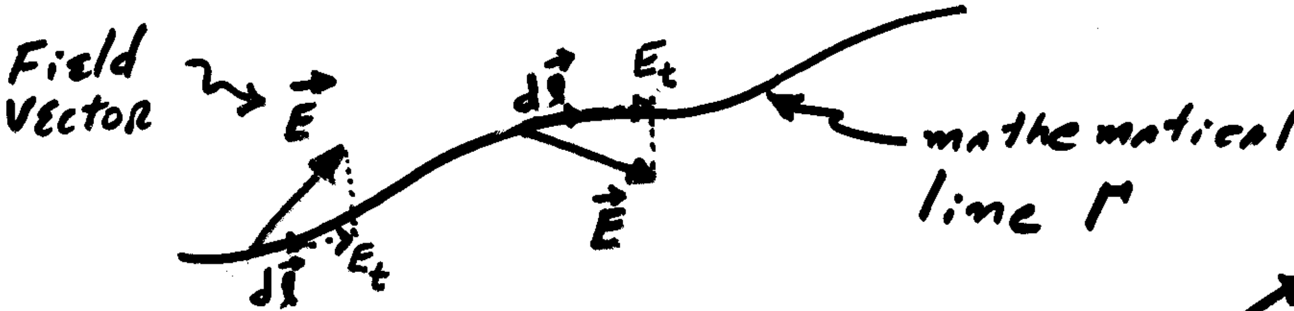
operators GRADIENT
DIVERGENCE
ROTATIONAL

Gauss' theorem

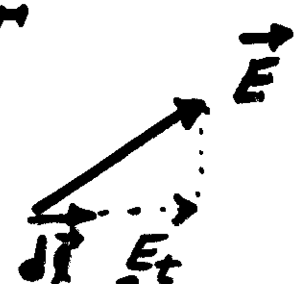
Stoke's theorem

GENERATION, PROPAGATION and
DETECTION of ELECTROMAGNETIC WAVES

Line integral

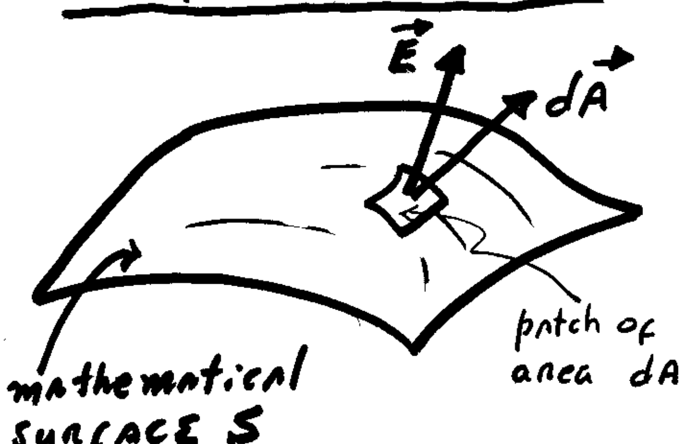


$$\int_\Gamma E_t \, dl = \int_\Gamma \vec{E} \cdot d\vec{l}$$



tangential component of $\vec{E}$ along $d\vec{l}$

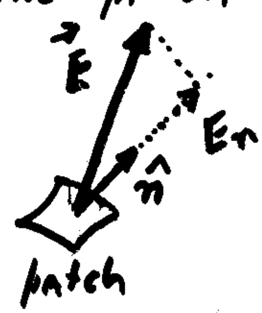
Surface integral



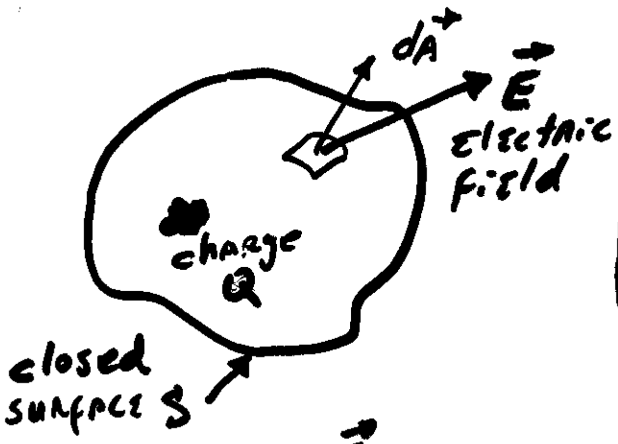
$$d\vec{A} = \hat{n} \, dA$$

where
$\hat{n}$ is the unit vector perpendicular to the patch

$$\int_S E_n \, dA = \int_S \vec{E} \cdot \hat{n} \, dA = \int_S \vec{E} \cdot d\vec{A}$$

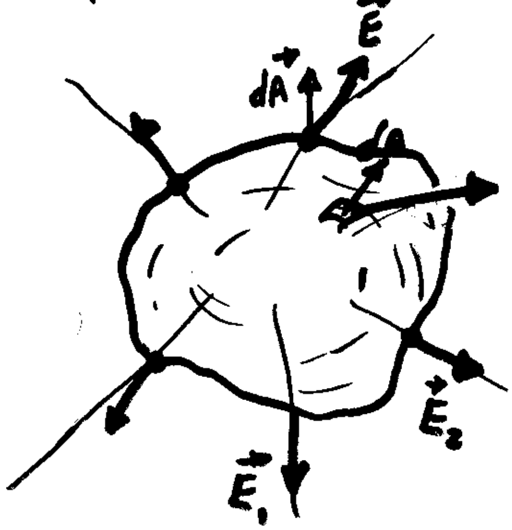


First Maxwell Eq



$$\boxed{\int_S \vec{E} \cdot d\vec{A} = \frac{Q_{inside}}{\epsilon_0}}$$

ϵ_0: permitivity of free-space



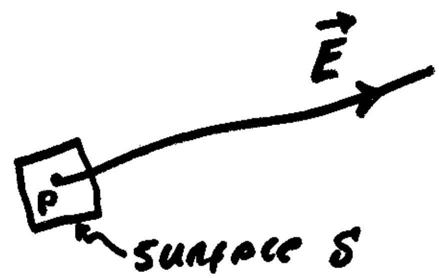
$\vec{E}$ is evaluated at each point ON the surface S

electric flux $\Phi_E = \int_S \vec{E} \cdot d\vec{A}$

Notice:

If an electric field line originates at a given point P,
then
there must exist charge at that point

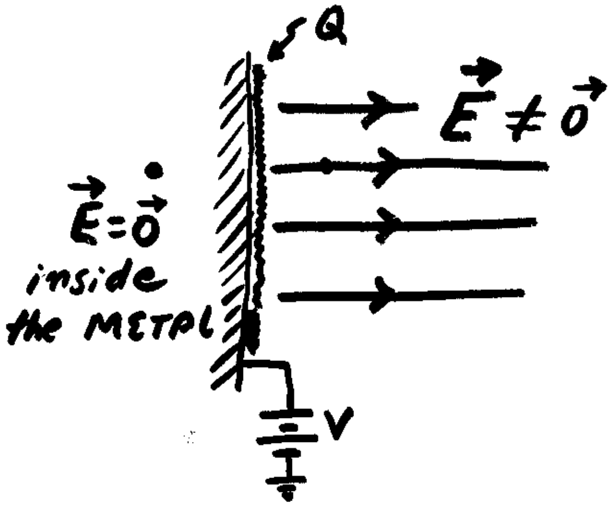
Proof



obviously $\Phi_E \neq 0$

Therefore, according to the 1st ME, there must be charge at P

This is what happens in metals, for example.

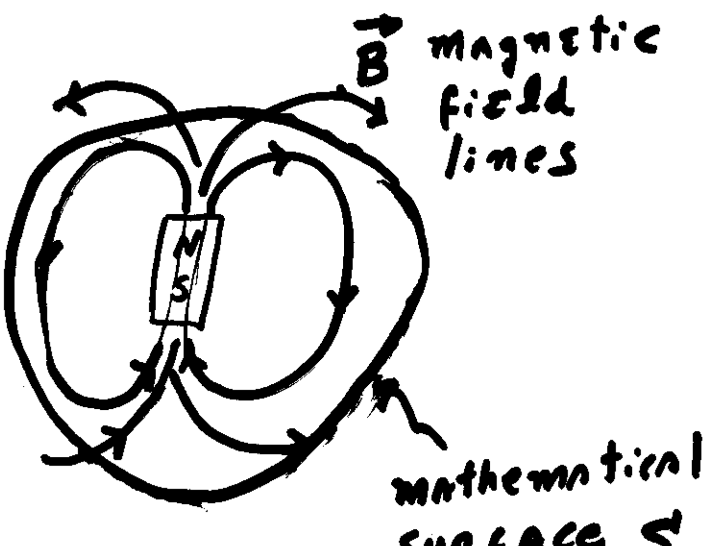


When charges are present:

lines of $\vec{E}$ originate
on positive charges
and terminate
on negative charges

Everywhere else
the $\vec{E}$ lines can twist
and turn in space, but
they cannot start or
stop

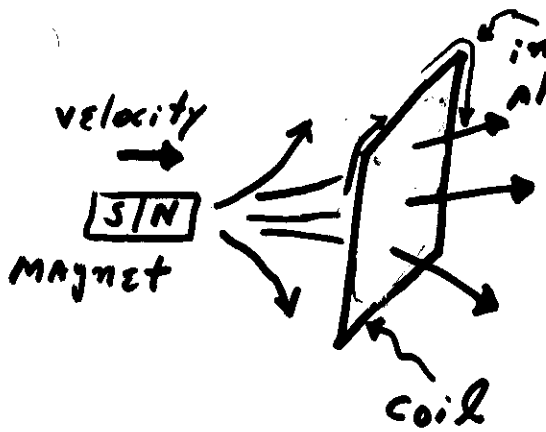
SECOND MAXWELL Eq.



$$\boxed{\int_S \vec{B} \cdot d\vec{A} = 0}$$

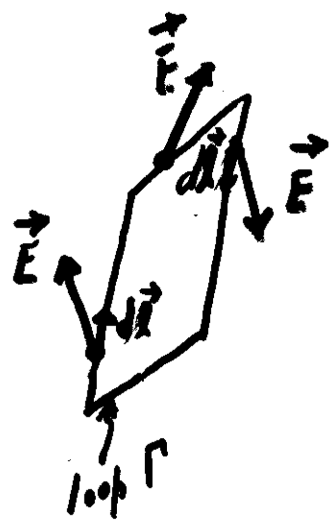
No magnetic monopoles
have been observed
(so far)

Third Maxwell Eq



The motion of the magnet produces an induced current along the coil

The existance of a current implies the presence of a electric field or (equivalently) an electromotive force



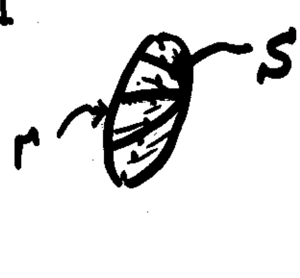
$$\underset{\text{electromotive force}}{\mathcal{E}} = \int_{\substack{\text{loop}\\ \Gamma}} \vec{E} \cdot d\vec{l} \quad (= Ri)$$

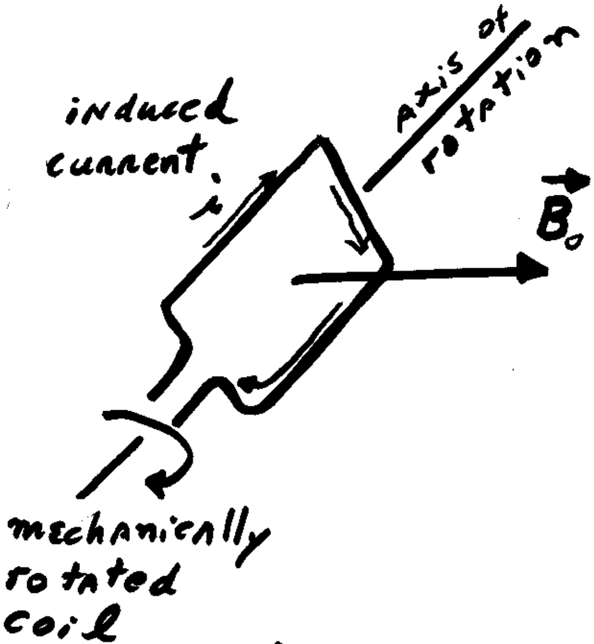
But, what is the value of $\mathcal{E}$?
(1 V, 2 V, −2.3 V … ?)

$$\boxed{\mathcal{E} = \int_{\substack{\text{loop}\\ \Gamma}} \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\left(\int_S \vec{B} \cdot d\vec{A}\right)}$$

where S is any open surface having the loop Γ as its boundary

$$\mathcal{E} = -\frac{d}{dt}\Phi_M$$

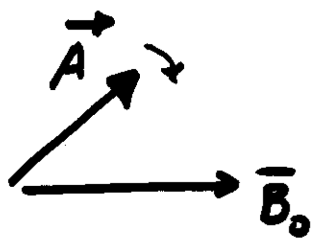




Notice:

A $\vec{B} = \vec{B}(t)$ ensures the induction of an electric field $\vec{E}$ and, consequently, a current i along a coil.

However, a static magnetic field $\vec{B}_0$ can also be exploited to generate currents along a coil (see figure).



$\vec{A}$ revolves around $\vec{B}_0$ causing a time dependent magnetic flux Φ_M

So, to have an electromotive force what matters is to have time dependent $\Phi_M(t)$

[In the figure above, since there is not an induced electric field, how can we have an electromotive force $\mathcal{E} = \int \vec{E} \cdot d\vec{s}$? Shouldn't $\vec{E}$ be zero because $\vec{E} = \vec{0}$?

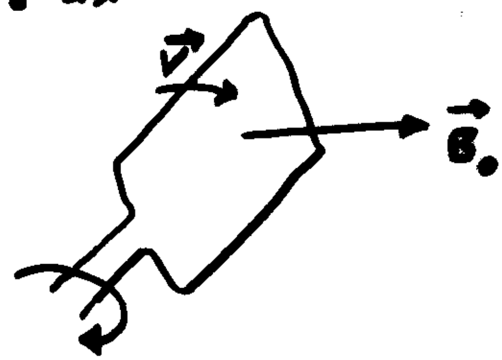
Answer: We better re-define $\mathcal{E}$ as

$$\mathcal{E} = \int_{\text{loop } \Gamma} \frac{\overset{\text{Magnetic}}{\text{Electric}} \text{ force on } q}{q} \cdot d\vec{l}$$

For the particular case of a mechanically rotated coil immersed in a uniform magnetic field $\vec{B}_0$ (see previous figure)

$$\mathcal{E} = \int_{\text{loop } \Gamma} \frac{\text{magnetic force on } q}{q} \cdot d\vec{l}$$

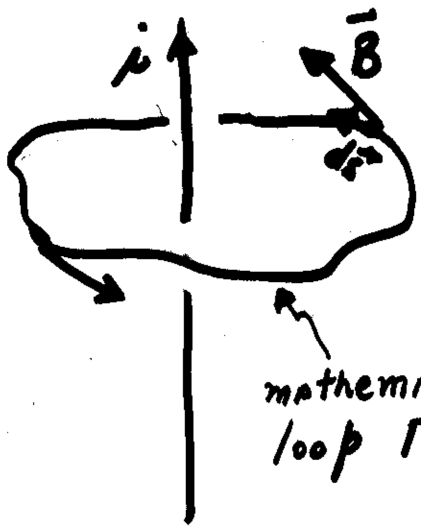
$$= \int_{\text{loop } \Gamma} \frac{q\,\vec{v} \times \vec{B}_0}{q} \cdot d\vec{l}$$



$$= \int_{\text{loop } \Gamma} (\vec{v} \times \vec{B}_0) \cdot d\vec{l} \quad \Big]$$

4th Maxwell's Eq.

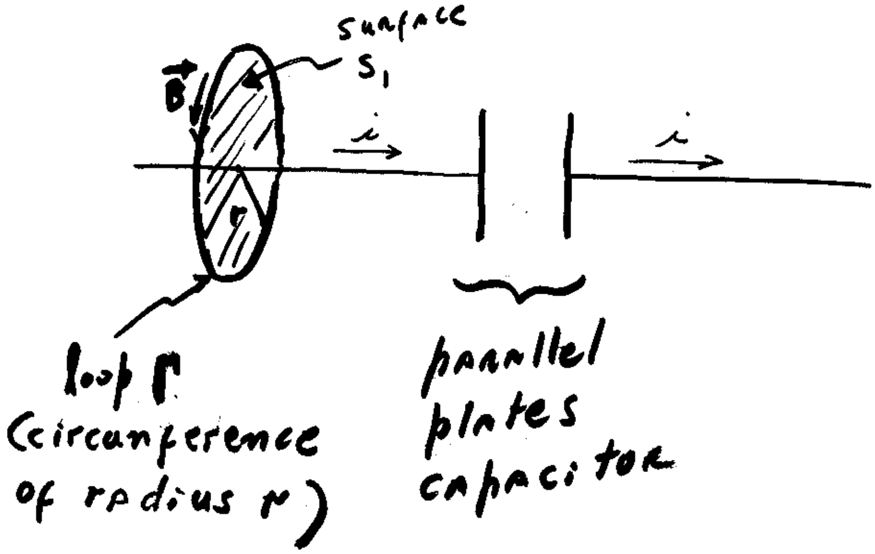
It built upon Ampere's Law



$$\int_{\text{loop } \Gamma} \vec{B} \cdot d\vec{s} = \mu_0 \underbrace{i}_{\text{current enclosed by the loop } \Gamma}$$

Maxwell notice this law was incomplete

James Clerk Maxwell noticed there was something wrong with the fourth equation. For example:



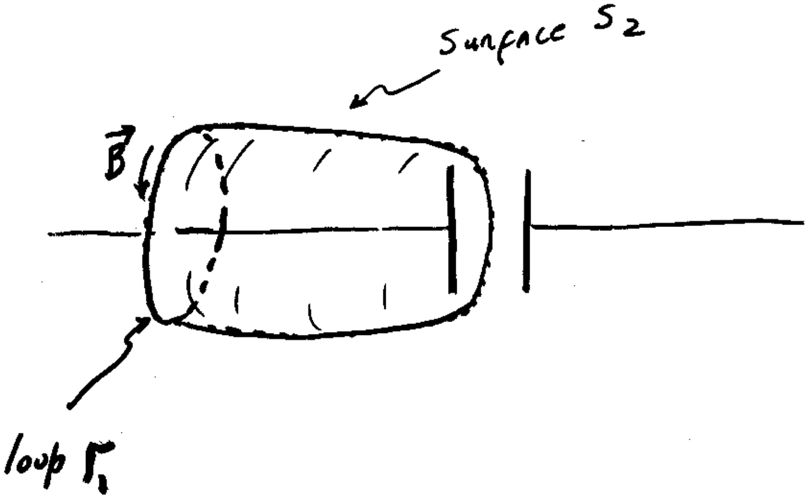
$$\int_\Gamma \vec{B}\cdot d\vec{l} = \mu_0 \underbrace{i}_{\text{current crossing the surface } S_1}$$

$$B\, 2\pi r = \mu_0 i$$

$$\Rightarrow$$

So, $B = \dfrac{\mu_0}{2\pi r} i$ ①

On the other hand,



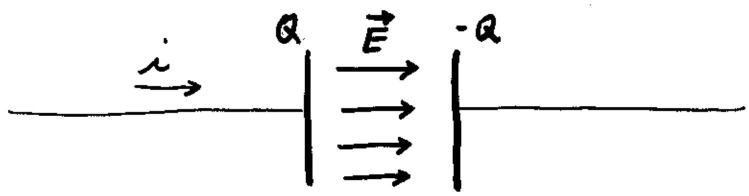
$$\int_\Gamma \vec{B}\cdot d\vec{l} = \mu_0 \underbrace{0}_{\text{current crossing the surface } S_2}$$

$$B\, 2\pi r = \mu_0 \times 0 \quad ②$$

So, $B = 0$

How can be possible that ① and ② give different results for the same B?
Something must be wrong with the Ampere's law

In order to solve this contradictory situation, let's take a look to what is going on inside the parallel plates capacitor

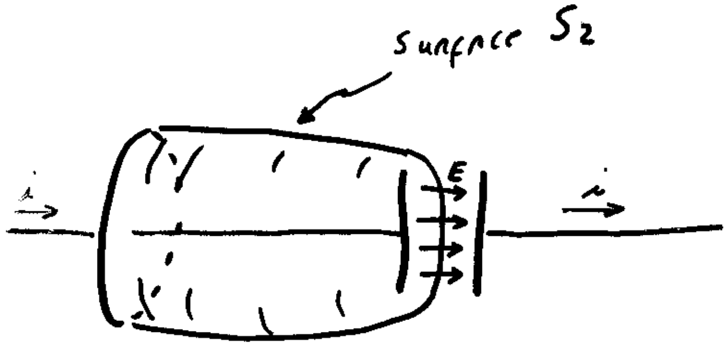


Let's remember

$$E = \frac{Q}{A\,\epsilon_0}$$

(A is the area of the plates)

The current i continuously accumulates charge on the plates, this is $Q = Q(t)$



Notice, the electric flux ϕ_E crossing the surface S_2 is given by

$$\phi_E = EA = \frac{Q}{\epsilon_0}$$

or

$$Q = \epsilon_0\,\phi_E$$

From the last expression, we obtain

$$\underbrace{\frac{dQ}{dt}}_{\text{but, this is equal to } i} = \epsilon_0 \frac{d\phi_E}{dt}$$

$$\boxed{i = \epsilon_0 \frac{d\phi_E}{dt}}$$

So, an electric flux that changes with time is "equivalent" to a current

Maxwell proposed the following modified version of the Ampere's law

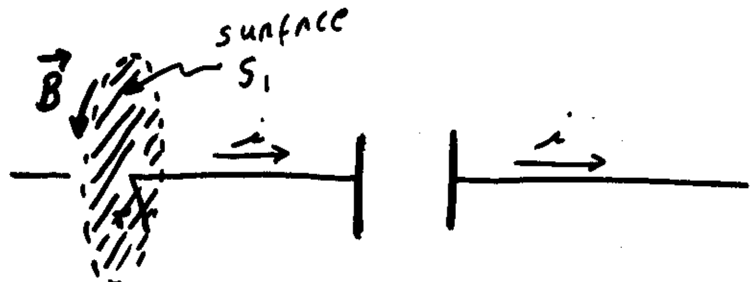
$$\int_{\text{loop } \Gamma} \vec{B}\cdot d\vec{s} = \mu_0 i + \underbrace{\mu_0 \epsilon_0 \frac{d\Phi_E}{dt}}_{\text{called "displacement current" } i_d}$$

or, written more explicitly

$$\boxed{\int_{\text{loop } \Gamma} \vec{B}\cdot d\vec{l} = \mu_0 \underbrace{i}_{\text{current enclosed by } \Gamma} + \mu_0 \epsilon_0 \frac{d}{dt}\int_{\text{surface } S} \vec{E}\cdot d\vec{A}}$$

Fourth Maxwell equation (1873)

Going back to our example, let's apply the 4th Maxwell equation and find $\vec{B}$ using the surface S_1



loop Γ

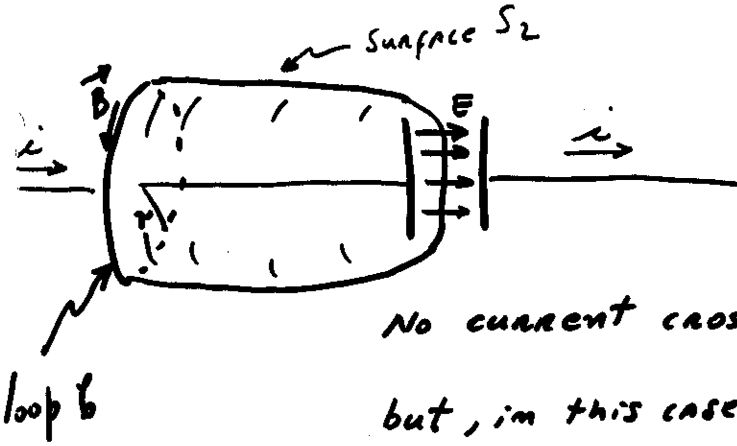
No electric field crossing the surface S_1, so the 4th Maxwell equation takes the form:

$$\int_{\text{loop } \Gamma} \vec{B}\cdot d\vec{l} = \mu_0 i$$

From which we obtain

$$B = \frac{\mu_0 i}{2\pi r}$$

What if we choose the surface S_2?



No current crossing the surface S_2, but, in this case, there does exist a electric field $\vec{E}$ crossing the surface. So, the 4th Maxwell equation takes the form:

$$\int_{\text{loop } \Gamma} \vec{B}\cdot d\vec{l} = \mu_0 \epsilon_0 \frac{d}{dt} \underbrace{\int_{\substack{\text{surface}\\ S_2}} \vec{E}\cdot d\vec{A}}_{= EA = \frac{Q}{A\epsilon_0}A = \frac{Q}{\epsilon_0}}$$

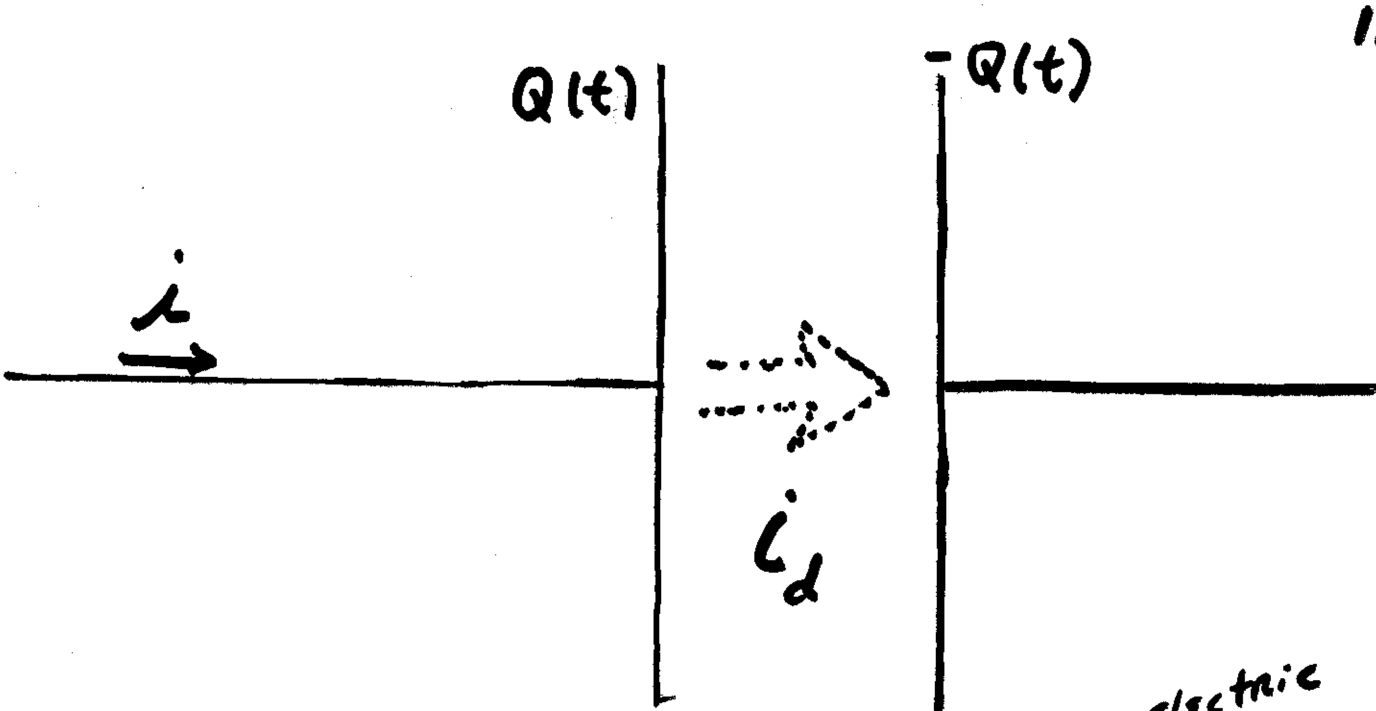
$$= \mu_0 \epsilon_0 \frac{d}{dt}\left(\frac{Q}{\epsilon_0}\right) = \mu_0 \underbrace{\frac{dQ}{dt}}_{} = \mu_0 i$$

$$\int_{\text{loop } \Gamma} \vec{B}\cdot d\vec{l} = \mu_0 i$$

From which we obtain

$$B = \frac{\mu_0}{2\pi r} i$$

Same result as 4 when the surface S_1 was used.



$$i = \frac{dQ}{dt}$$

charge on the plate

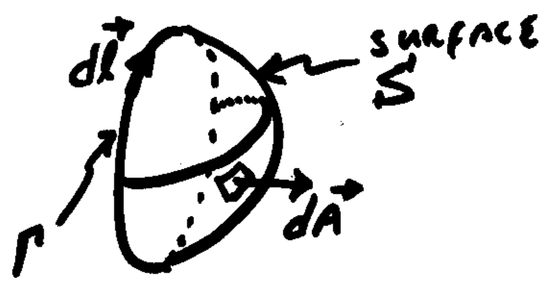
$$i_d = \epsilon_0 \frac{d}{dt} \Phi_E$$

electric flux

"displacement current"

More general,

$$\int_\Gamma \vec{B} \cdot d\vec{l} = \mu_0 \int_S \vec{j} \cdot d\vec{A} + \mu_0 \epsilon_0 \frac{d}{dt} \int_S \vec{E} \cdot d\vec{A}$$



4th MAXWELL's equation

Maxwell Equations in differential form. –

First, some definitions in vector algebra:

- The operator ∇ (called "gradient")

$$\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

- ∇ acts on scalar fields

Example

If ϕ is the electric potential,

$$\nabla \phi = \left(\frac{\partial}{\partial x}\phi, \frac{\partial}{\partial y}\phi, \frac{\partial}{\partial z}\phi \right)$$

which, as we know, gives

$$= (-E_x, -E_y, -E_z)$$

Thus,

$$\nabla \phi = -\vec{E}$$

We see, when ∇ acts on a scalar field it gives a vector

- The divergence operator "$\nabla\cdot$"

Given a vector field $\vec{E}$,

$$\nabla\cdot\vec{E} \equiv \frac{\partial}{\partial x}E_x + \frac{\partial}{\partial y}E_y + \frac{\partial}{\partial z}E_z$$

$\nabla\cdot\vec{E}$ is a scalar quantity

- The rotational operator "$\nabla\times$"

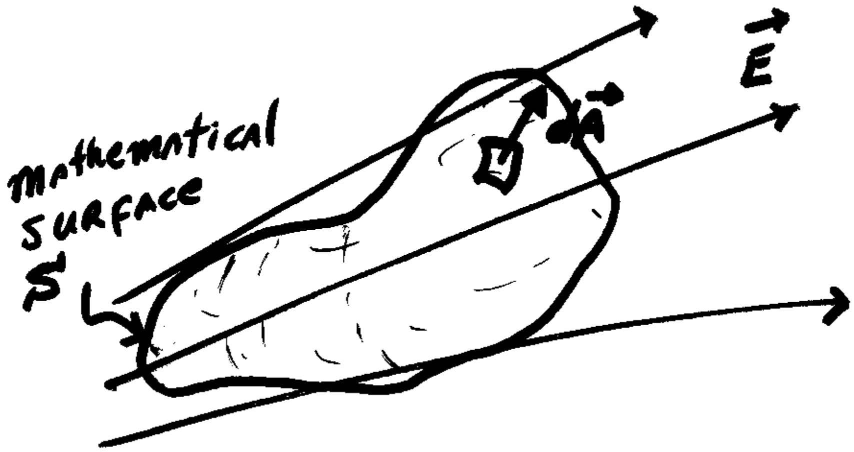
Given a vector field $\vec{E}$,

$$\nabla\times\vec{E} \equiv \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}E_z - \frac{\partial}{\partial z}E_y, \frac{\partial}{\partial z}E_x - \frac{\partial}{\partial x}E_z, \frac{\partial}{\partial x}E_y - \frac{\partial}{\partial y}E_x \right)$$

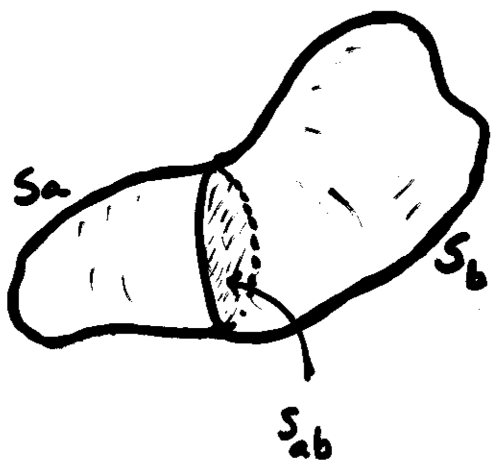
$\nabla\times\vec{E}$ is a vectorial quantity

• About the flux of a vector field



$$\Phi = \int_S \vec{E} \cdot d\vec{A}$$

Flux through the surface S



Now, surface S has been divided into two contiguous surfaces S_1 and S_2

$S_1 = S_a + S_{ab}$

$S_2 = S_b + S_{ab}$

Notice:

Flux through S =

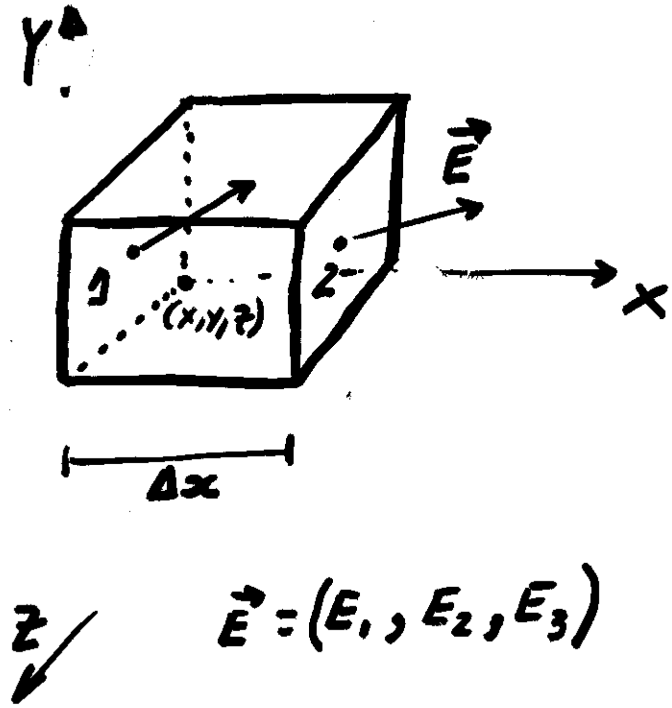
= flux through S_1

+

flux through S_2

$$\Phi_S = \Phi_{S_1} + \Phi_{S_2}$$

GAUSS' theorem



$$\vec{E} = (E_1, E_2, E_3)$$

What is the flux through the cube of volume $\Delta x \Delta y \Delta z$?

$$\phi_{cube} = \phi_1 + \phi_2 + \ldots + \phi_6$$

$$\phi_1 = -\underbrace{E_1(x)}\,\Delta y \Delta z$$

$$\phi_2 = \underbrace{E_1(x+\Delta x)}\,\Delta y \Delta z$$

$$\phi_1 + \phi_2 = \left[E_1(x+\Delta x) - E_1(x)\right]\Delta y \Delta z$$

$$= \left[\frac{\partial E_1}{\partial x}\Delta x\right]\Delta y \Delta z$$

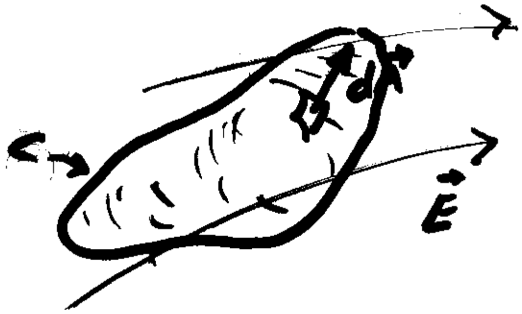
Similar expressions for $\phi_3 + \phi_4$ and $\phi_5 + \phi_6$

Thus,

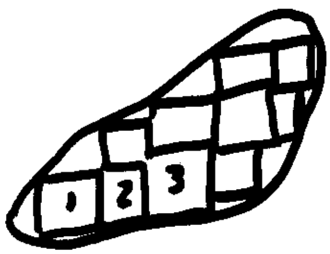
$$\phi_{cube} = \left[\frac{\partial E_1}{\partial x} + \frac{\partial E_2}{\partial y} + \frac{\partial E_3}{\partial z}\right]\Delta x \Delta y \Delta z$$

$$= \left[\nabla \cdot \vec{E}\right]\Delta V$$

flux through the cube of volume $\Delta V = \Delta x \Delta y \Delta z$



$$\Phi_S = \int_S \vec{E} \cdot d\vec{A}$$ surface integral



Notice, the volume inside the surface S can be diveded into many very small cubes

(we just need to make them very tiny; infinitesimal)

Each cube_i has a surface A_i and a volume ΔV_i

We know that

$$\Phi_S = \Phi_{S_1} + \Phi_{S_2} + \cdots$$

$$= (\nabla \cdot \vec{E})\Delta V_1 + (\nabla \cdot \vec{E})\Delta V_2 + \cdots$$

which is nothing but

$$= \int_V \nabla \cdot \vec{E}\, dv$$ ← a volume integral

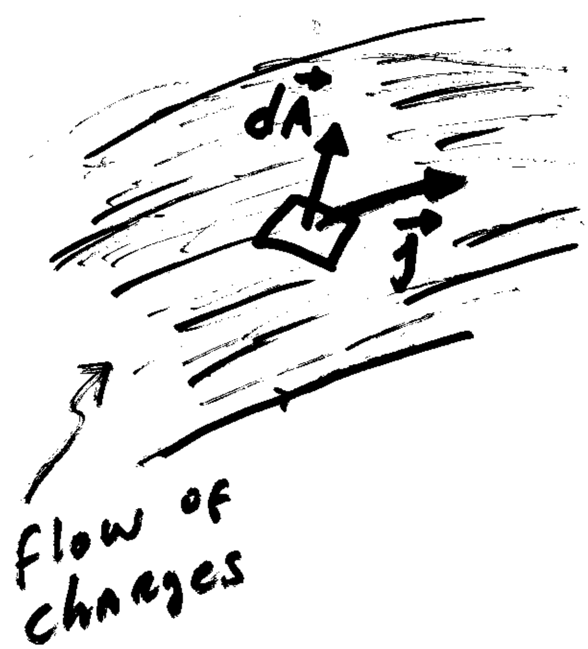
Thus, combining this expression with the one above, we have obtained

$$\int_S \vec{E} \cdot d\vec{A} = \int_V \nabla \cdot \vec{E} \, dv$$ GAUSS theorem

S is any mathematical closed surface

V is the volume inside S

Example: Expressing the charge conservation principle in differential form

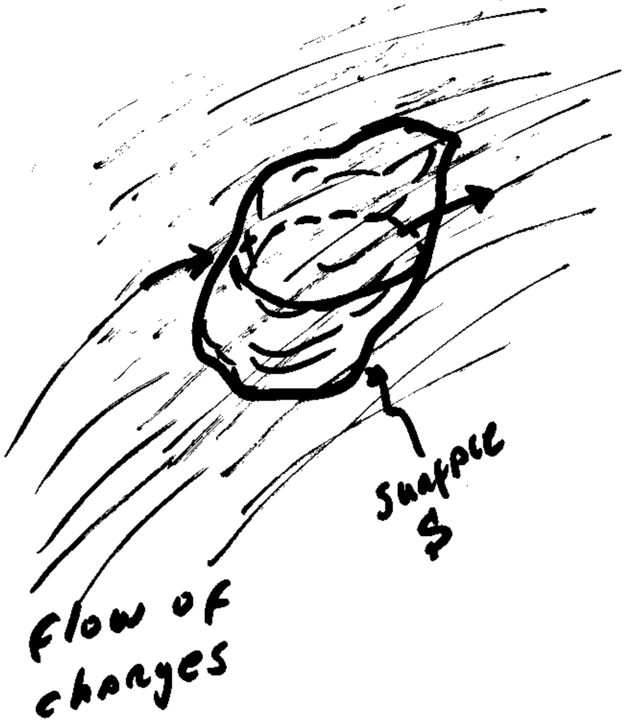


$$j = \frac{\text{current}}{\text{area}}$$

$\vec{j}$ points in the direction of the local velocity $\vec{v}$ of the charges

$$\text{current crossing } d\vec{A} = \vec{j} \cdot d\vec{A}$$

in units of coulomb/sec



$I =$ net current crossing the surface S

$$= \int_S \vec{j} \cdot d\vec{A}$$

I must be equal to the charge per unit time of the net charge lost inside the volume of the surface S

$$I = -\frac{d}{dt} Q_{inside}$$

$$= -\frac{d}{dt} \int_V \rho \, dv$$

thus,

$$\underbrace{\int_S \vec{j} \cdot d\vec{A}}_{\text{using Gauss' theorem}} = -\frac{d}{dt} \int_V \rho \, dv$$

where ρ is the charge density (coulomb/m^3)
$\rho = \rho(x, y, z, t)$

$$\int_S \nabla \cdot \vec{j} \, dV = -\frac{d}{dt} \int_V \rho \, dv \underset{\substack{\text{since the} \\ \text{surface } S \text{ is} \\ \text{stationary}}}{\xrightarrow{=}} -\int_V \frac{\partial \rho}{\partial t} \, dv$$

Since the surface S is arbitrary (it could even be infinitesimal), it must be that

$$\boxed{\nabla \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0}$$ change conservation in differential form

Example First Maxwell Equation

$$\underbrace{\int_S \vec{E} \cdot d\vec{A}}_{\substack{\text{applying} \\ \text{Gauss theorem}}} = \frac{Q_{inside}}{\epsilon_0} = \frac{1}{\epsilon_0} \int_V \rho \, dv$$

$$\int_V \nabla \cdot \vec{E} \, dv = \int_V \frac{\rho}{\epsilon_0} \, dv$$

Since this is valid for any volume, including an infinitesimal one, it must be that

$$\boxed{\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}}$$ First's Maxwell Eq in differential form

Example Second Maxwell's Eq

$$\int_S \vec{B} \cdot d\vec{A} = 0$$

Applying Gauss theorem

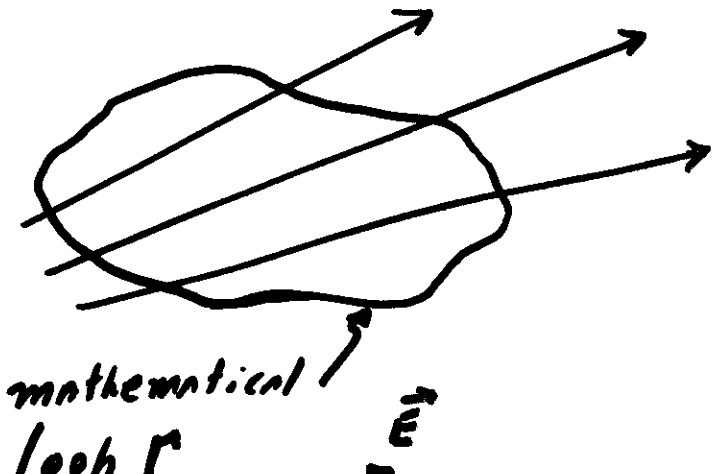
$$\int_V \nabla \cdot \vec{B} \, dv = 0$$ for any arbitrary volume V

This implies

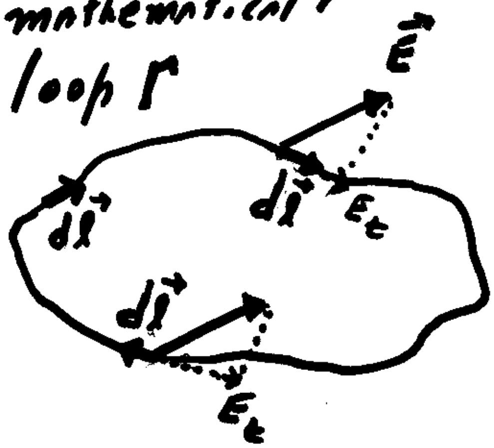
$$\boxed{\nabla \cdot \vec{B} = 0}$$ Second Maxwell's Eq in differential form

Stoke's theorem

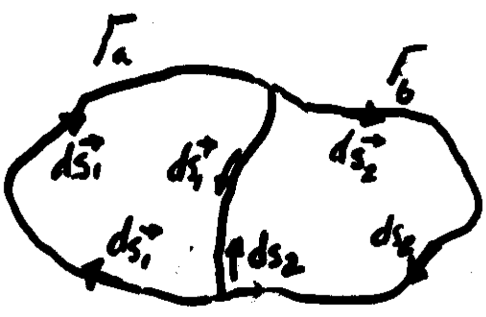
About the circulation of a vector field $\vec{E}$



Circulation of $\vec{E}$ is the line integral of the tangential component of $\vec{E}$ around the loop Γ



$$\int E_t \, dl = \int \vec{E} \cdot d\vec{l}$$



Now, the loop Γ has been divided into two contiguous loops Γ_1 and Γ_2

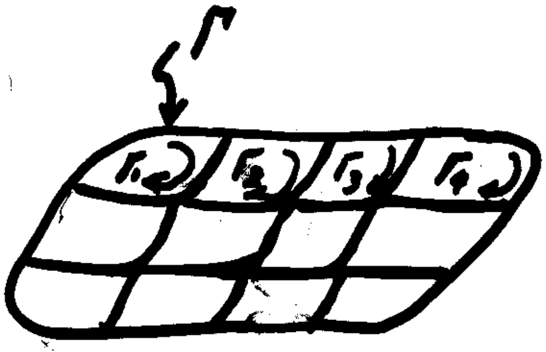
Notice:

$$\int_{\Gamma} = \int_{\Gamma_1} + \int_{\Gamma_2}$$

$$\Gamma_1 = \Gamma_a + \Gamma_{ab}$$

$$\Gamma_2 = \Gamma_b + \Gamma_{ab}$$

Generalization



$$\int_{\Gamma} = \int_{\Gamma_1} + \int_{\Gamma_2} + \ldots$$

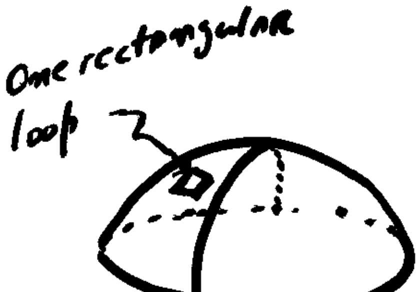
Here we have assumed that the different loops $\Gamma_1, \Gamma_2, \ldots$ are in the plane of Γ.

But it doesn't have to be that way

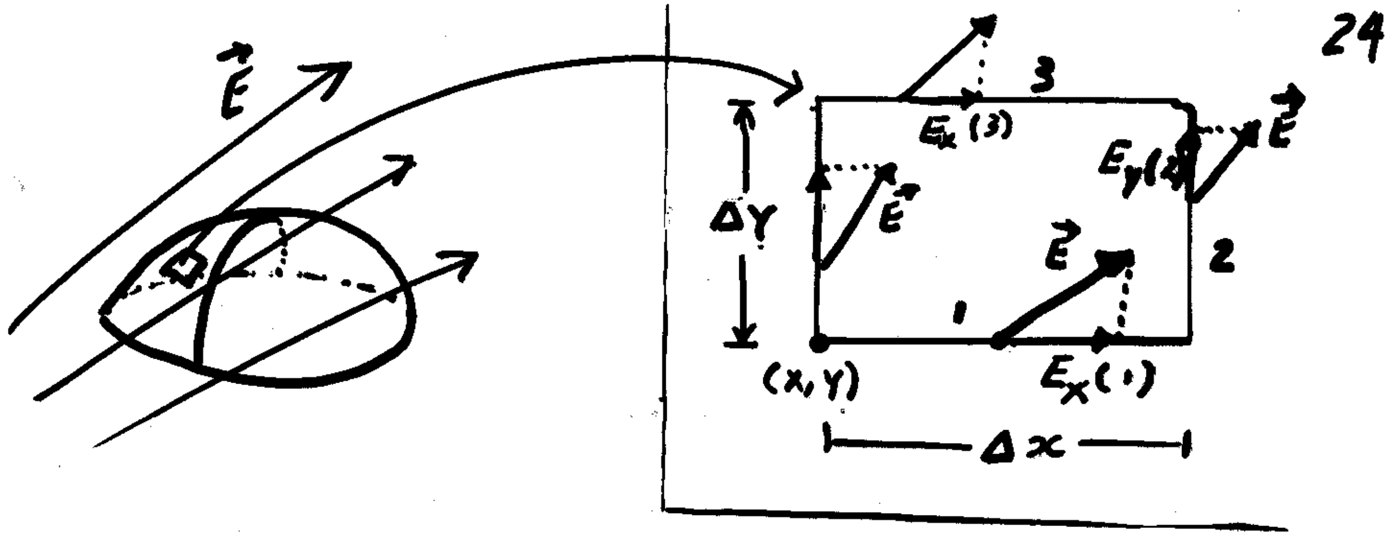
The small loops can lie on any surface having Γ as its boundary

By choosing the loops small enough, each can be considered a flat rectangular loop.

Circulation around a rectangular loop



For the particular small rectangular loop of area $\Delta x \Delta y$ shown in the figure, let's choose a reference such that the loop results lying at the XY plane



If the result of the circulation can later be put in vectorial notation, then it will be the same no matter how the axis XYZ were chosen

$$\int_{\square} \vec{E} \cdot d\vec{s} = E_x(1)\,\Delta x + E_y(2)\,\Delta Y + - E_x(3)\,\Delta x - E_y(4)\,\Delta Y$$

Since $E_x(3) - E_x(1) \approx \dfrac{\partial E_x}{\partial Y} \Delta Y$

$$E_y(2) - E_y(4) \approx \frac{\partial E_y}{\partial x} \Delta x$$

$$\int_{\square} \vec{E} \cdot d\vec{s} = \underbrace{\left(\frac{\partial E_Y}{\partial x} - \frac{\partial E_x}{\partial Y} \right)}_{\text{z-component of } \nabla \times \vec{E}} \Delta x\, \Delta Y$$

$$\oint_{\square} \vec{E} \cdot d\vec{s} = \left(\nabla \times \vec{E}\right)_z \, \Delta x \, \Delta y$$

We can express this result in vectorial form by realizing that z component is the direction perpendicular to the small loop

$$= \left(\nabla \times \vec{E}\right)_n \Delta x \Delta y = \left(\nabla \times \vec{E}\right) \cdot \hat{n} \, \Delta x \, \Delta y$$

$$\oint_{\square} \vec{E} \cdot d\vec{s} = \left(\nabla \times \vec{E}\right) \cdot d\vec{A}$$

$d\vec{A} = \hat{n} \, \Delta x \Delta y$

We extend this result to multi-connected loops

$$\boxed{\oint_{\Gamma} \vec{E} \cdot d\vec{l} = \int_S \left(\nabla \times \vec{E}\right) \cdot d\vec{A}}$$

Stoke's theorem

S is any surface whose boundary is Γ

Example: Third Maxwell Eq

$$\underbrace{\int_\Gamma \vec{E}\cdot d\vec{l}}_{\text{Applying Stoke's theorem}} = \underbrace{-\frac{d}{dt}\int_S \vec{B}\cdot d\vec{A}}_{\text{if the surface } S \text{ is stationary}}$$

$$\int_S (\nabla \times E)\cdot d\vec{A} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}$$

this expression is valid for ~~any~~ arbitrary surface, including a infinitesimal rectangle. Thus

$$\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

Third Maxwell Eq in differential form

Example: Fourth Maxwell's Eq

$$\underbrace{\int_\Gamma \vec{B}\cdot d\vec{l}}_{\text{Applying Stoke's theorem}} = \mu_0 \int_S \vec{j}\cdot d\vec{A} + \epsilon_0 \mu_0 \underbrace{\frac{d}{dt}\int_S \vec{E}\cdot d\vec{A}}_{\text{If } S \text{ is stationary}}$$

$$\int_S (\nabla \times \vec{B})\cdot d\vec{A} = \mu_0 \int_S \vec{j}\cdot d\vec{A} + \epsilon_0 \mu_0 \int_S \frac{\partial \vec{E}}{\partial t}\cdot d\vec{A}$$

Since this expression is valid for any arbitrary surface S, then

$$\boxed{\nabla \times \vec{B} = \mu_0 \vec{j} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}}$$

4rth Maxwell's Eq
in differential form