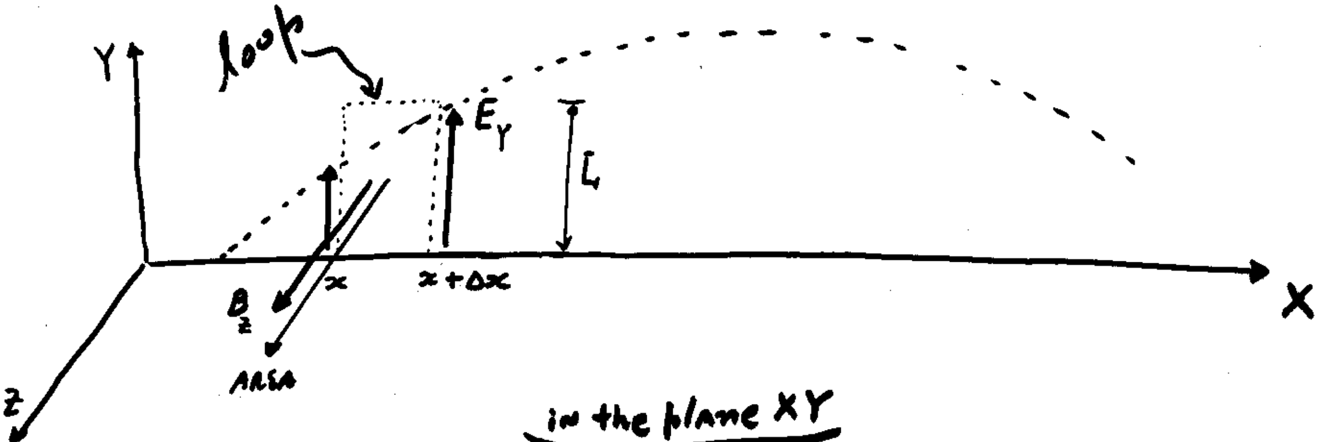
Selfsustained propagation of electromagnetic waves



Let's consider a $\underbrace{\text{loop}}_{}$ (in the plane XY) of width $= \Delta x$ and height $= L$,

and apply Faraday's law:

$$\mathcal{E} = -\frac{d}{dt}\Phi_m$$

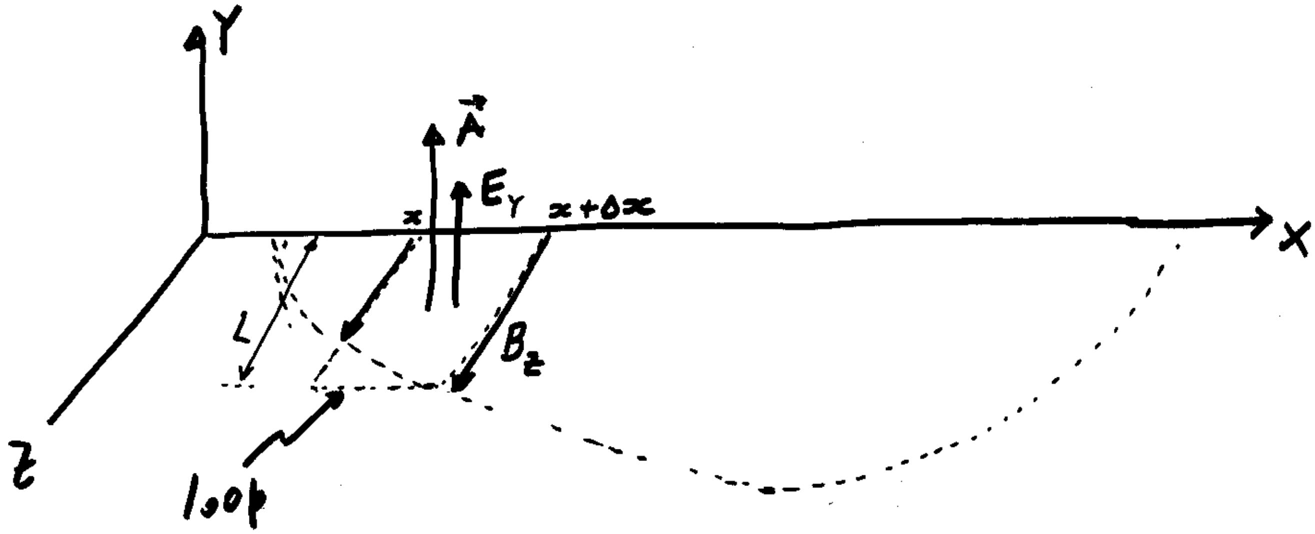
$$\mathcal{E} = \int_{loop} \vec{E}\cdot d\ell = L\,\underbrace{E_Y(x+\Delta x)} - \underbrace{L\,E_Y(x)}$$

$$\Phi_m = B_z A = B_z L\,\Delta x$$

$$L\left(E_Y(x+\Delta x) - E_Y(x)\right) = -L\,\Delta x\,\frac{\partial B_z}{\partial t}$$

$$\frac{E_Y(x+\Delta x) - E_Y(x)}{\Delta x} = -\frac{\partial B_z}{\partial t}$$

$$\Rightarrow \quad \frac{\partial E_Y}{\partial x} = -\frac{\partial B_z}{\partial t} \qquad (1)$$



Let's consider a loop of width Δx and length $= L$, and apply the modified Ampere's law (the one that includes the displacement current).

$$\underbrace{\int_{loop} \vec{B} \cdot d\vec{l}} = \underbrace{\mu_0 i}_{=0} + \mu_0 \epsilon_0 \underbrace{\frac{d}{dt} \phi_E}$$

$= 0$ because there is no current in the free space

$$= \frac{d}{dt} E_Y A = L \Delta x \frac{\partial E_Y}{\partial t}$$

$$L B_z(x) - L B_z(x+\Delta x)$$

Thus, we obtain:

$$- \frac{B_z(x+\Delta x) - B_z(x)}{\Delta x} = \frac{\partial E_Y}{\partial t} \Rightarrow \frac{\partial B_z}{\partial x} = -\mu_0 \epsilon_0 \frac{\partial E_Y}{\partial t} \quad (2)$$

Let's combine ① and ②

From ① $\Rightarrow$ $$\frac{\partial^2 E_Y}{\partial x^2} = -\frac{\partial^2 B_z}{\partial x\, \partial t}$$

From ② $\Rightarrow$ $$\frac{\partial^2 B_z}{\partial t\, \partial x} = -\mu_0 \epsilon_0 \frac{\partial^2 E_Y}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 E_Y}{\partial x^2} - \mu_0 \epsilon_0 \frac{\partial^2 E_Y}{\partial t^2} = 0$$

The resulting equation can be written as

$$\boxed{\frac{\partial^2 E_Y}{\partial x^2} - \frac{1}{\left(\frac{1}{\mu_0 \epsilon_0}\right)} \frac{\partial^2 E_Y}{\partial t^2} = 0}$$

Equation to be satisfied by the electric field E_Y

When compared with the wave equation

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} = 0$$

which has solutions of the form $y = y(x \pm vt)$

we realize that:

> The ELECTRIC FIELD E_Y constitutes a wave that propagates with velocity
> $$v = \sqrt{\frac{1}{\mu_0 \epsilon_0}}$$

this is,

$$E_y = E_y(x \pm vt)$$

* EXERCISE: • Use equations ① and ② to obtain

$$\frac{\partial^2 B_z}{\partial x^2} - \frac{1}{\left(\frac{1}{\mu_0 \epsilon_0}\right)} \frac{\partial^2 B_z}{\partial t^2} = 0$$

- Verify that, in particular, the following wave

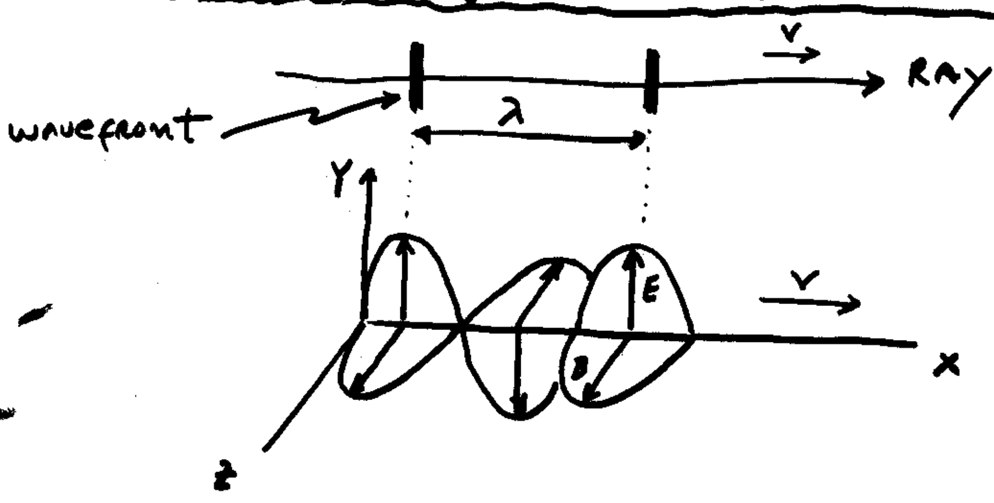
  $B_z = B_z(x - vt)$, where $v = \sqrt{1/\mu_0 \epsilon_0}$

  satisfies the equation above.

- Given the wave

  $$B_z = \underbrace{1.6 \times 10^{-7}\,\text{Tesla}}_{\text{const}} \cos(kx - wt)$$

  what should be the value of w/k if we are told that B_z is the magnetic field of a propagating electromagnetic wave.



Pictorial representation of a sinusoidal electromagnetic wave

Do we know the values of ϵ_0 and μ_0 ?

Yes, indeed.

$$\epsilon_0 = 8.85 \times 10^{-12} \frac{F}{m} \qquad \mu_0 = 1.26 \times 10^{-6} \, H/m$$

Let's calculate the velocity of an electromagnetic wave

$$\epsilon_0 \mu_0 = 11.1 \times 10^{-18} \; s^2/m^2$$

$$\frac{1}{\epsilon_0 \mu_0} = 0.09 \times 10^{18} \; m^2/s^2$$

$$\sqrt{\frac{1}{\epsilon_0 \mu_0}} = 0.3 \times 10^9 \, m/s = 300{,}000 \; km/s$$ the speed of light !

With excitement Maxwell wrote:

"We can scarcely avoid the inference that light consists in the transverse undulation of the same medium which is the cause of electric and magnetic phenomena"

Notation $\quad c = 300{,}000 \; km/s$

Summary

MAXWELL EQUATIONS in VACUUM

$$\int_{\text{surface}} \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}$$

$$\int_{\text{surface}} \vec{B} \cdot d\vec{A} = 0$$

$$\int_{\text{loop}} \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_{\text{surf}} \vec{B} \cdot d\vec{A}$$

$$\int_{\text{loop}} \vec{B} \cdot d\vec{l} = \mu_0 \int \vec{j} \cdot d\vec{A} + \mu_0 \epsilon_0 \frac{d}{dt} \int_{\text{surf}} \vec{E} \cdot d\vec{A}$$

E: Electric field

B: magnetic field

$\vec{j}$: current density

$$E = E(t) \xrightarrow[\text{produces}]{} B$$

$$B = B(t) \longrightarrow E$$

$$\frac{\partial^2 E}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = 0 \qquad \frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} = 0$$

$$c = \sqrt{\frac{1}{\epsilon_0 \mu_0}} \qquad c = 300{,}000 \text{ km/s}$$

$$E = E(x \pm ct) \qquad B = B(x \pm ct)$$