APPLIED OPTICS

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9.2.A Planes

Let $\vec{\mathbf{r}} = (x,y,z)$ and $\hat{\mathbf{n}}$ be the spatial coordinates and a unit vector, respectively.

Notice,

$$\vec{\mathbf{r}} \cdot \hat{\mathbf{n}} = const$$

locates the points $\vec{\mathbf{r}}$ that constitute a plane perpendicular to $\hat{\mathbf{n}}$.

Different planes are obtained when using different values for the constant value (as seen in the figure below).

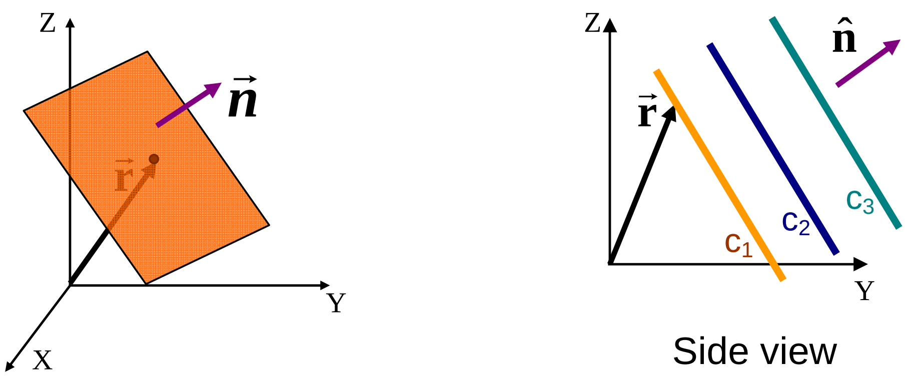


***Fig. 9.5. Left:** A plane perpendicular to the unit vector* $\hat{\mathbf{n}}$. ***Right:** Different planes are obtained when using different values for the constant value c in the expression* $\vec{\mathbf{r}} \cdot \hat{\mathbf{n}} = c(const)$.

9.2.B Traveling Plane Waves and Phase velocity

Consider the two-variable function ψ of the form

$$\psi(\mathbf{r},t) = f(\underbrace{\vec{\mathbf{r}} \cdot \hat{\mathbf{n}} - \mathrm{v}t}_{phase})$$

where f is an arbitrary one-variable function (f and $\hat{\mathbf{n}}$ become determined once a specific problem is considered).

Notice, the points over a plane oriented perpendicular to $\hat{\mathbf{n}}$ and traveling with velocity v define the locus of points that keep the phase of the wave ψ constant. For this reason, the wave ψ is called a plane wave.

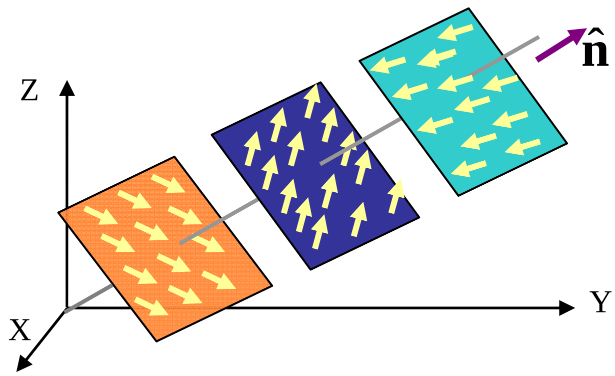


***Fig. 9.6** Schematic representation of a plane wave of electric fields. The figure shows the electric fields at three different planes, at a given instant of time. The fields lie oriented on the corresponding planes. All the planes are perpendicular to the unit vector* $\hat{\mathbf{n}}$.

Traveling Plane Waves (propagation in one dimension)

$f(x-\mathrm{v}t)$ For any arbitrary function f, this represents a wave propagating to the right with speed *v*.
f could be COS, EXP, ... etc.

$f(\underbrace{x-\mathrm{v}t}_{phase})$ Notice, a point x advancing at speed v will keep the phase of the wave f constant.
For this reason v is called the *phase velocity* v_{ph}.

Traveling Harmonic Waves

$$COS(kx-\omega t) = COS\ [k(x-\frac{\omega}{k}t)\ ] \quad \text{and} \quad e^{i(k\,x-\omega\,t)}$$

These are specific examples of waves propagating to the right with phase velocity $\mathrm{v}_{ph} = \omega / k$.

In general $\omega = \omega\ (k)$.

That is, for different values of k, the corresponding waves travel with different phase velocities.

The specific relationship $\omega = \omega\ (k)$ depends on the specific physical system under analysis (waves in a crystalline array of atoms, light propagation, etc.

9.2.C A Traveling Wave-package and its Group Velocity

Consider the expression $f(x) = (\ 1/\sqrt{2\pi}\ )\int_{-\infty}^{\infty} F(k)\ e^{ikx} dk$ as the representation of a pulse profile at $t{=}0$. Here e^{-ikx} is the profile of the harmonic wave $e^{i(kx-\omega\, t)}$ at $t = 0$. The profile of the pulse at a later time will be represented by,

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\ \ e^{i(k\,x-\omega\, t)} F(k)$$

Pulse decomposed into a group of traveling harmonic waves (37)

Since, in the general case, each component of the group travels with its own phase velocity,

would still it possible to associate a unique velocity to the propagating group of waves?

The answer is positive; it is called *group velocity*. Below we present an example that helps to illustrate this concept.

Case: Wavepacket composed of two harmonic waves

Analytical description)

For simplicity, let's consider the case in which the packet of waves consists if only two waves of very similar wavelength and frequencies.

$$\psi(x,t) = Cos[kx - \omega t] + Cos[\ (k+\Delta k)x - (\omega + \Delta\omega)t] \quad (38)$$

Using the identities $Cos(A+B)\ = Cos(A)Cos(B)\ -\ Sin(A)Sin(B)$

and $Cos\,(\ A-B)\ = Cos(A)\,Cos(B)\ +\ Sin(A)\,Sin(B)$ one obtains

$Cos(\ A+B)\ + Cos(\ A-B) = 2Cos(A)Cos(B)$, which can be expressed as

$$Cos(A) + Cos(B) = 2\,Cos(\frac{A+B}{2})\,Cos(\frac{A-B}{2})$$

Accordingly, (38) can be expressed as,

$$\psi(x,t) = 2Cos[\ (\frac{\Delta k}{2})x - (\frac{\Delta\omega}{2})t\ ]\,Cos[\ (k + \frac{\Delta k}{2})x - (\omega + \frac{\Delta\omega}{2})t\ ]$$

Since we are assuming that $\Delta\omega << \omega$ and $\Delta k << k$, we have

$$\psi(x,t) = \underbrace{2Cos[\ (\frac{\Delta k}{2})x - (\frac{\Delta\omega}{2})t\ ]}_{\text{Modulation envelope}}\,Cos[\ kx - \omega t\ ] \qquad (39)$$

Notice, the modulation envelope travels with velocity equal to

$$v_g = \frac{\Delta\omega}{\Delta k}, \qquad (40)$$

which is known as the group velocity.

In summary,

$$\psi(x,t) = \underbrace{2Cos[\ (\frac{\Delta k}{2})x - (\frac{\Delta\omega}{2})t\ ]}_{\text{Amplitude Modulating wave}}\ \underbrace{Cos[\ kx - \omega t\ ]}_{\text{Carrier wave}} \qquad (41)$$

Amplitude Modulating wave

Planes where the ***amplitude of the resultant wave remains constant*** *travel with speed* $V_g = \frac{\Delta\omega}{\Delta k}$

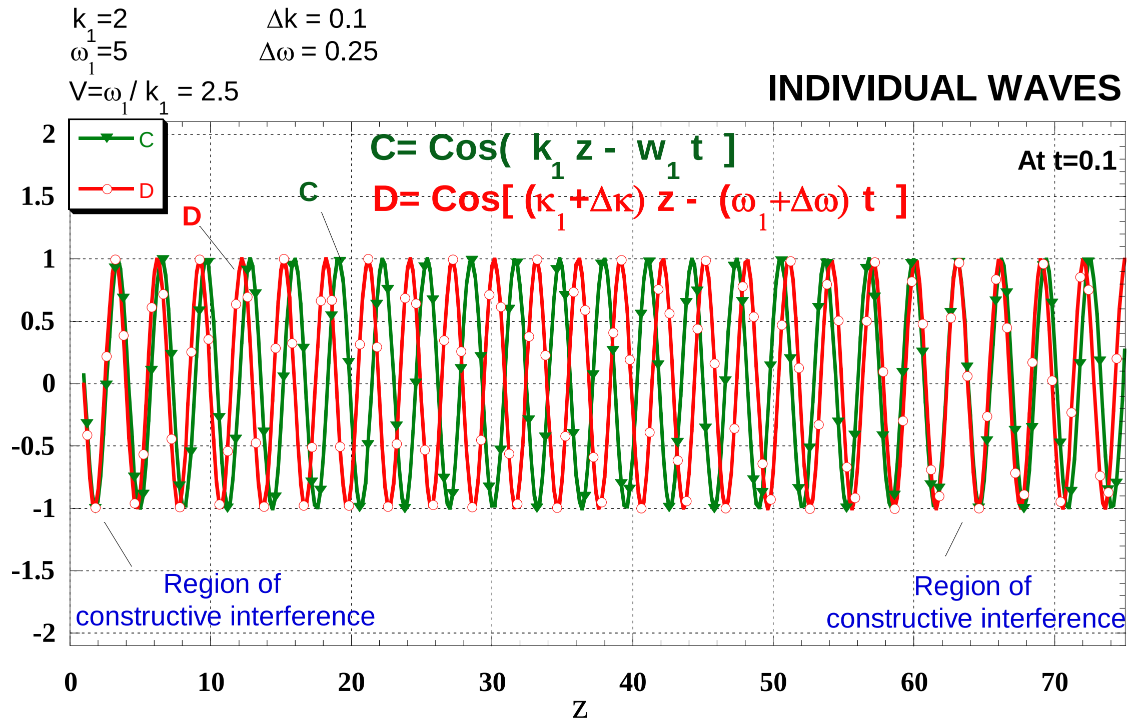
Carrier wave

Plane of ***constant phase*** *traveling with speed*

$$V_p = \frac{\omega}{k}$$

The *phase velocity* is a measure of the velocity of the harmonic waves components that constitute the wave.
The *group velocity* is the velocity at which the positions of maximum interference propagate

Graphical description



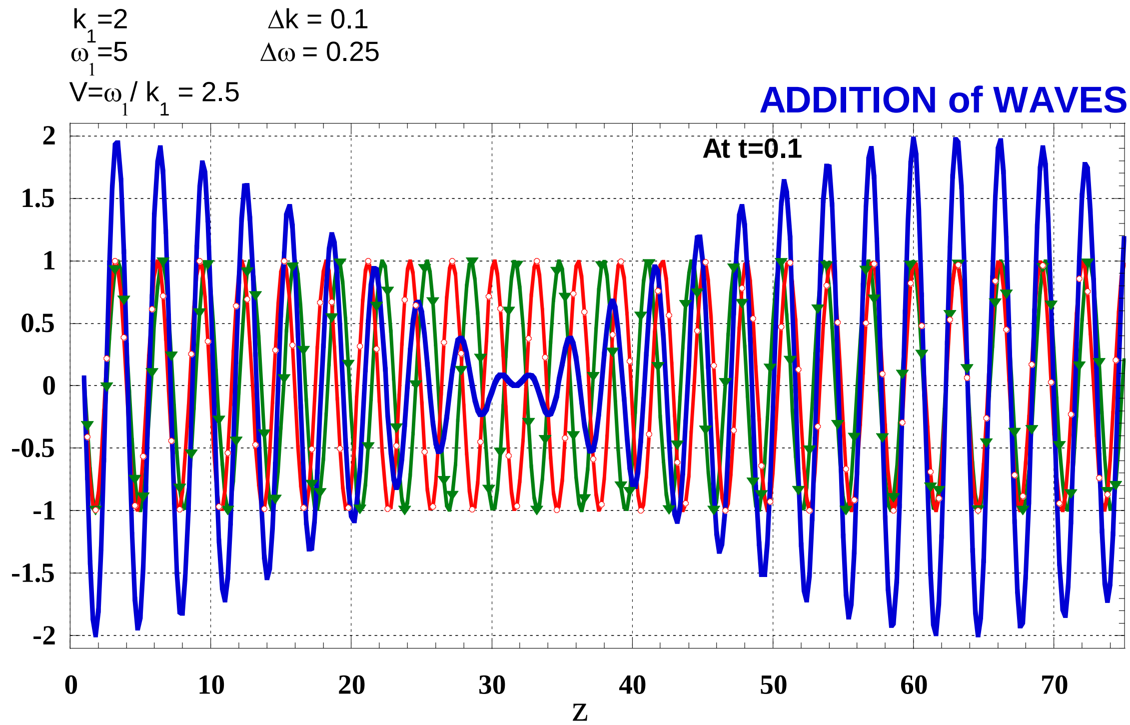


Fig. 9.7 *Addition of waves (phase velocity = group velocity = 2.5).*

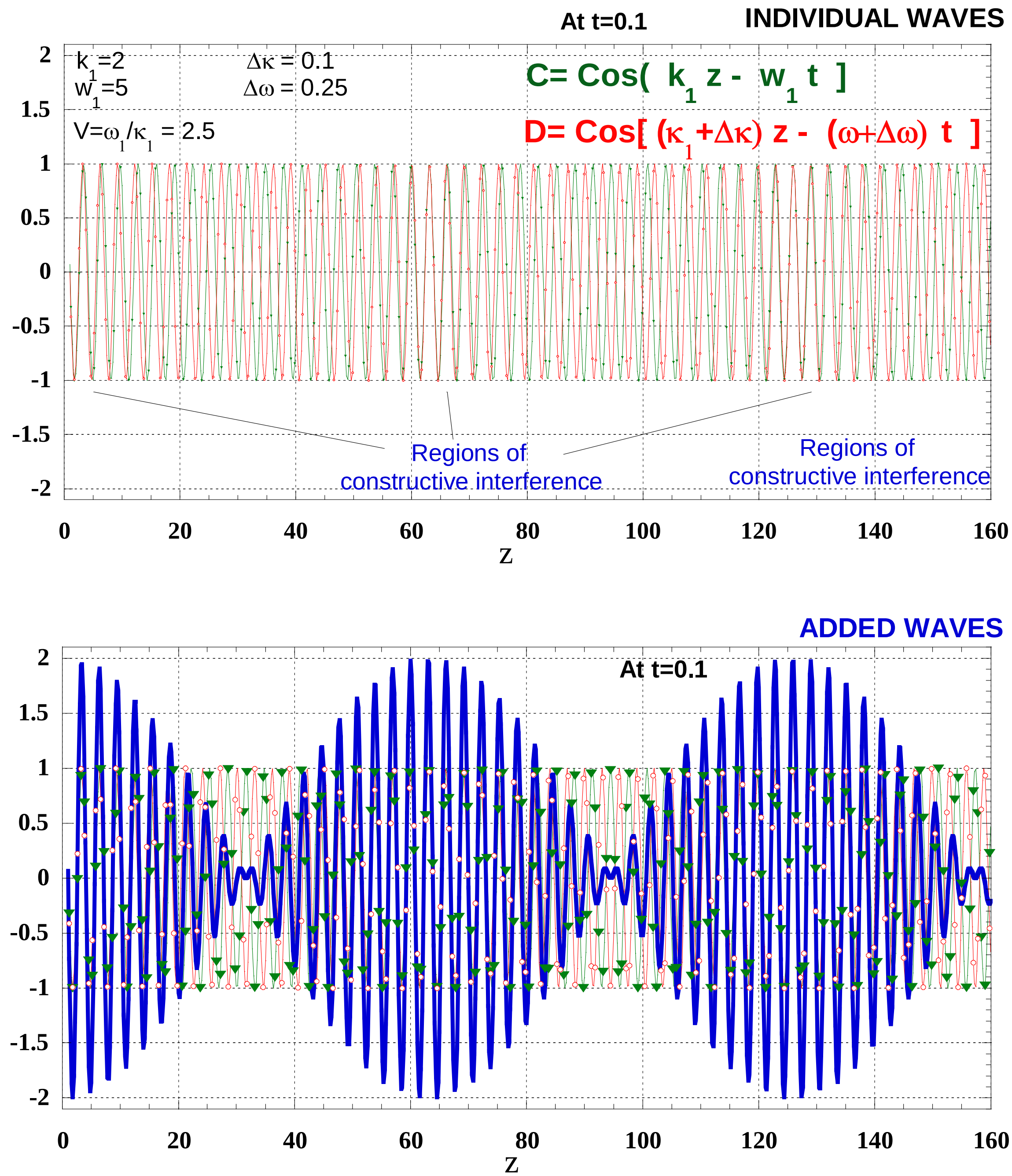


Fig. 9.8 *Addition of waves (phase velocity = group velocity = 2.5)..*

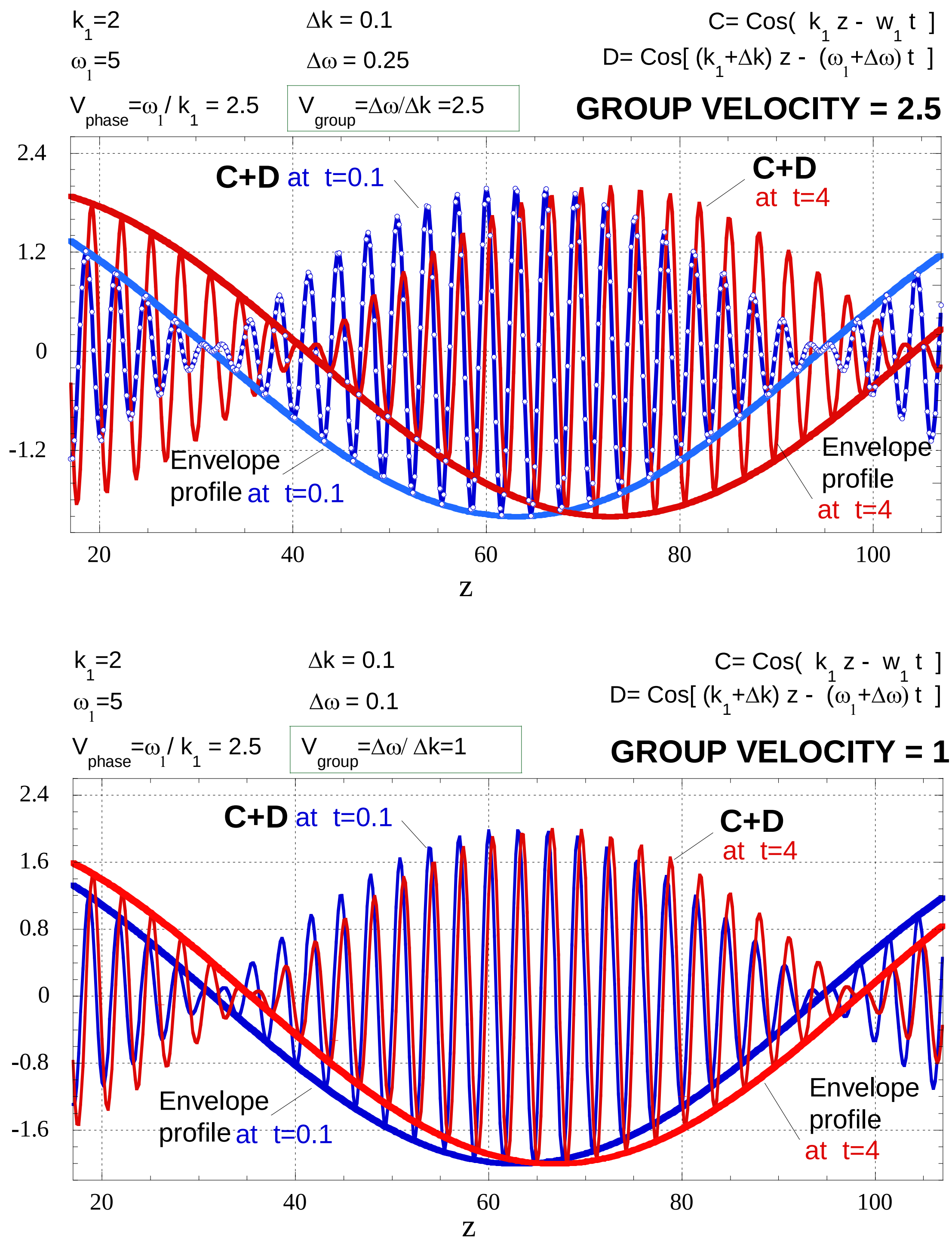


Fig. 9.9 *Wavepackets advancing at a sped equal to their group velocity, respectively.* ***Top:*** V_g*=2.5.* ***Bottom****:* V_g *=1. Notice, the result corroborates that the envelope profile of the lower group velocity advances less than the other one.*

Phasor method to analyze a wavepacket

It becomes clear from the analysis above that a packet composed of only two single harmonic waves of different wavelength can hardly represents a localized pulse. Rather, it represents a train of pulses. Let's try to understand qualitatively (using the mrthod of phasors) the reasons for the formation of a train of pulses.

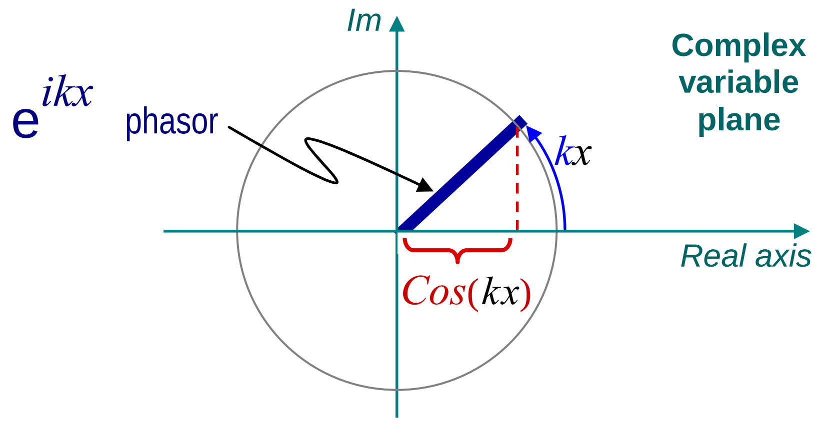


Fig. 9.10 *A phasor representation in the complex plane*

Case: wavepacket composed of two waves

Let's consider the addition of two harmonic waves

$$Cos(\underbrace{k_A x}_{phase}) \;+\; Cos(\underbrace{k_B x}_{phase}) \tag{42}$$

where $k_A < k_B$, and $k_B - k_A \equiv \Delta k$.

To evaluate (42) we will work in the complex plane. Accordingly, to each wave we will associate a corresponding phasor,

$$e^{ik_A x} + e^{ik_B x}$$

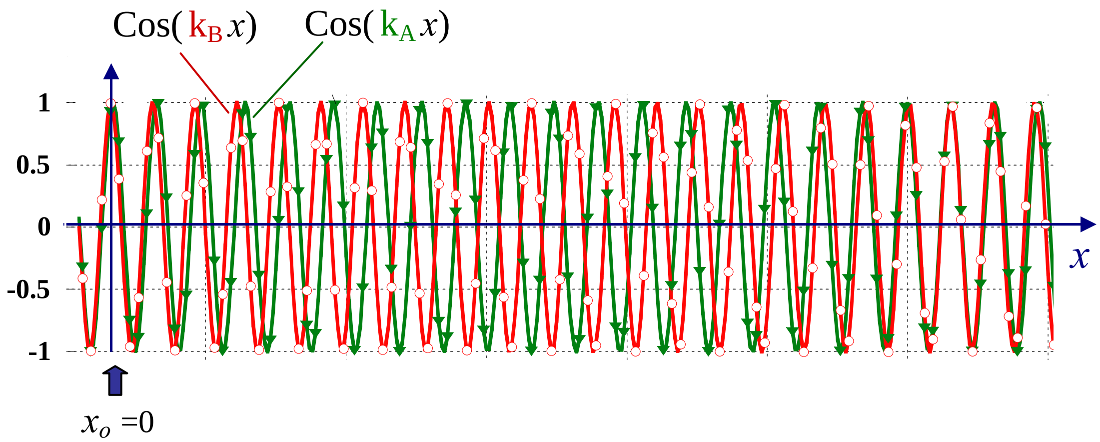
The projection of the (complex) phasors along the horizontal axis gives the real-value we are looking for in (42).

At a given position x, the phase-difference between the two wave profiles is equal to,

$$\textit{Phase difference} = k_B x - k_A x = (k_B - k_A)x \tag{43}$$

The following happens:

a) The waves interfere constructively at $x = 0 \equiv x_o$ (both waves have a phase equal to zero.)



b) As x increases a bit, the interference is not as perfect since the phase of the waves start to differentiate from each other $(k_B - k_A)x \neq 0$; consequently the sum of the waves should display an oscillatory behavior as x increases.

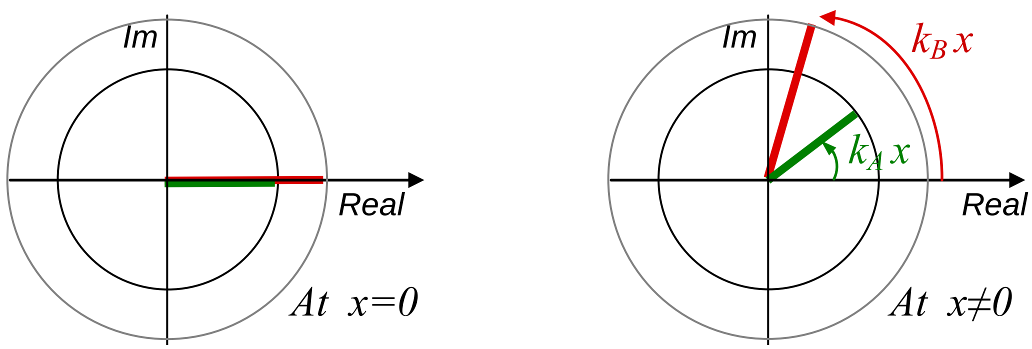


Fig. 9.11 *Analysis of wave addition by phasors in the complex plane. For clarity, the magnitude of one of the phasors has been drawn larger than the other one.*

c) As x increases, it will reach a particular value $x = x_1$ that makes the phase difference between the waves equal to 2π.

$$(k_B - k_A)x_1 = 2\pi$$

That is, the waves interfere constructively again at $x_1 = 2\pi / \Delta k$, where $k_B - k_A \equiv \Delta k$

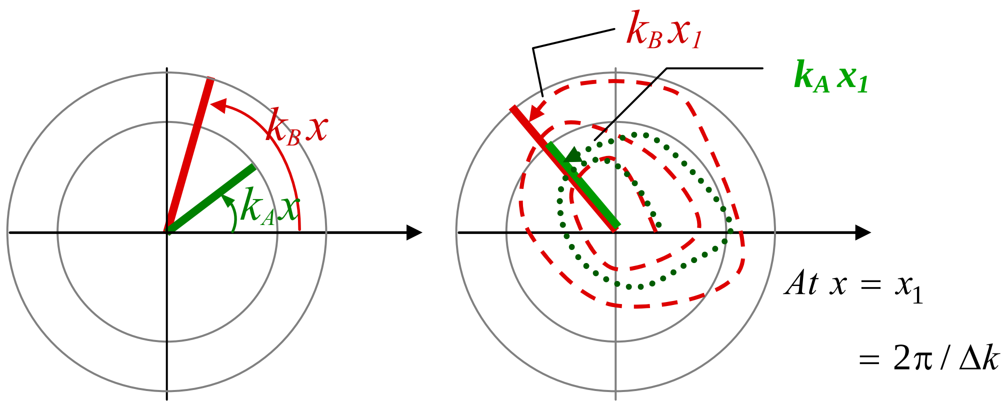
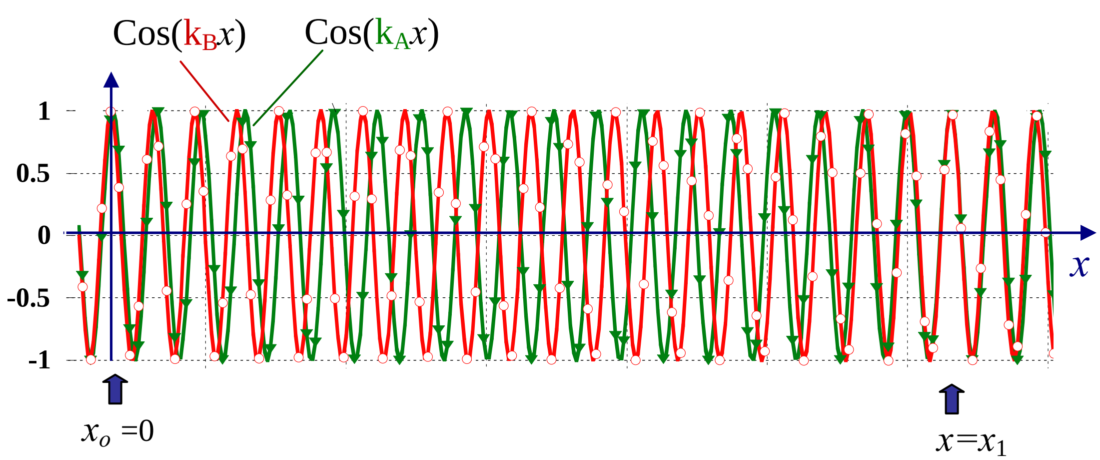


Fig. 9.12 Left: In general the phasors do not coincide. **Right:** At a specific value of $x=x_1$, both phasors coincide, thus giving a maximum value to the sum of the waves (at that location.) The phasors diagram also makes clear that as x keeps increasing, constructive interference will also occur at multiple values of x_1.

It is expected then that the wave-pattern (the sum of the two waves) observed around $x = 0$ will repeat again at around $x = x_1$.



d) Notice that additional regions of constructive interference will occur at positions $x = x_n$ satisfying $(k_B - k_A)x_n = n\ 2\pi$ or $x_n = n\ 2\pi / \Delta k$ (n = 1, 2, 3, …) The phasors diagram, therefore, makes clear that as x keeps increasing, additional discrete values (x_1, x_2, x_3, etc) will be found to produce additional maxima.

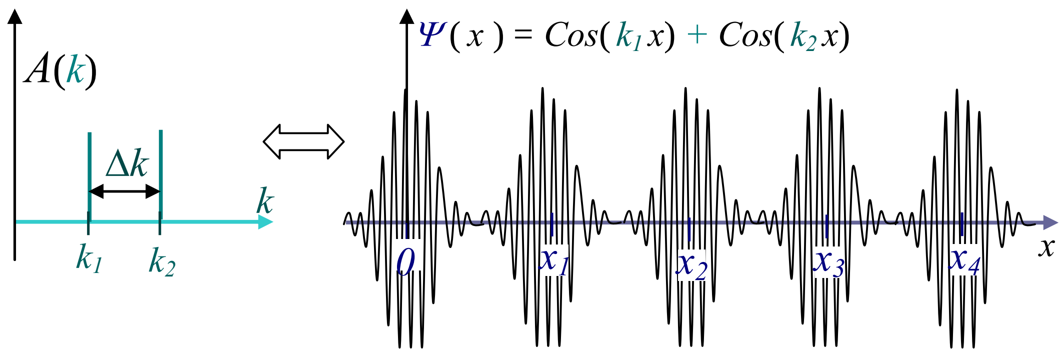


Fig. 9.13 Wavepacket composed of two harmonic waves

Case: A wavepacket composed of several harmonic waves

When adding several harmonic waves $\sum_{i=1}^{M} A_i Cos(k_i x)$, with M >2, the condition for having repeated regions of constructive interference still can occurs. In effect,

- First, there will be of course a constructive interference around x=0.
- Second, we expect the existence of a position $x=x_1$ that will make each of the quantities $(k_i-k_j)x_1$ equal to a multiple of 2π.

  $$(k_i-k_j)x_1 = (\text{integer})_{ij}\, 2\pi \qquad (44)$$

  (for all the ij combinations, with i and j = 1, 2, 3, … , M).

  When this happens, it would mean that all the corresponding phasors coincide, thus giving a maximum of amplitude.
- Third, additional regions of maximum interference will occur at multiple integers of x_1.

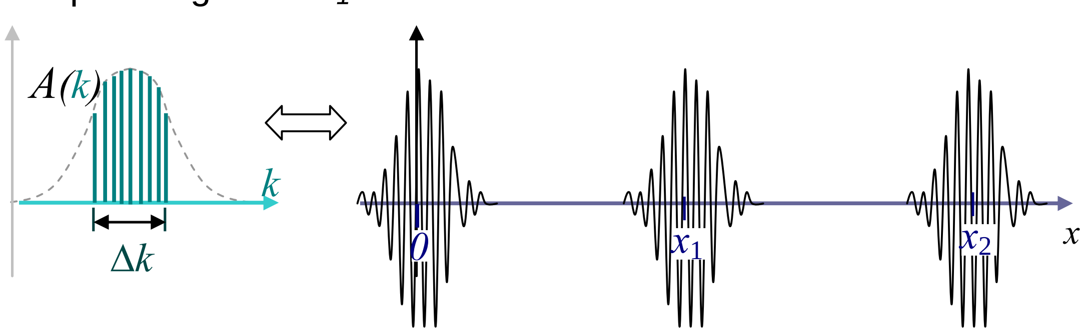


Fig. 9.14 *Wavepacket composed of a large number of harmonic*

waves (to be compared with Fig. 4.13 above.) The train of pulses are more separated from each other.

Notice also that,

the greater the number M of harmonic components in the packet (with wavevectors k_i within the same range Δk shown in the figure above),

the more stringent becomes for all the M waves to satisfy at once the condition (44) for constructive interference.

This means, a greater value of x may be needed any time an extra number of harmonic waves are included in the packet.

Since the other maxima of interference occur at multiple values of x_1, we expect, therefore, that the greater number of k-values (within the same range Δk), the more separated from each other will be the regions of constructive interference. This is shown in Fig. 4.14.

Case: Wavepacket composed of an infinite number of harmonic waves

Adding more and more wavevectors k (still all of them within the same range Δk show in the figure below)) will make the value of x_1 to become greater and greater. As we consider a continuum variation of k, the value of x_1 will become infinite. That is, we will obtain just one pulse.

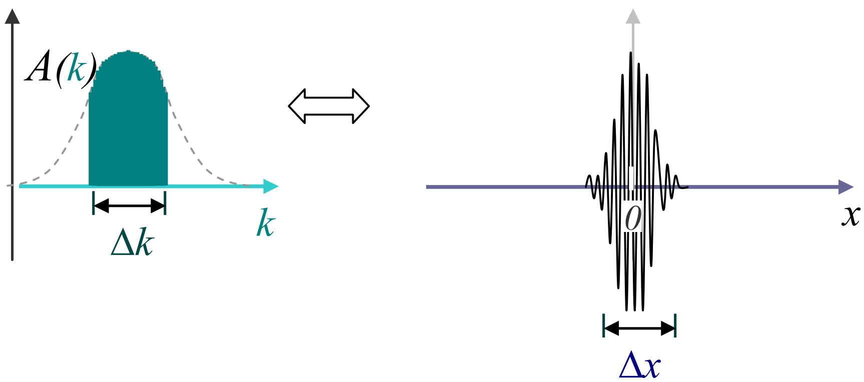


Fig. 9.15 *Wavepacket composed of wavevectors k within a continuum range Δk produces a single pulse.*

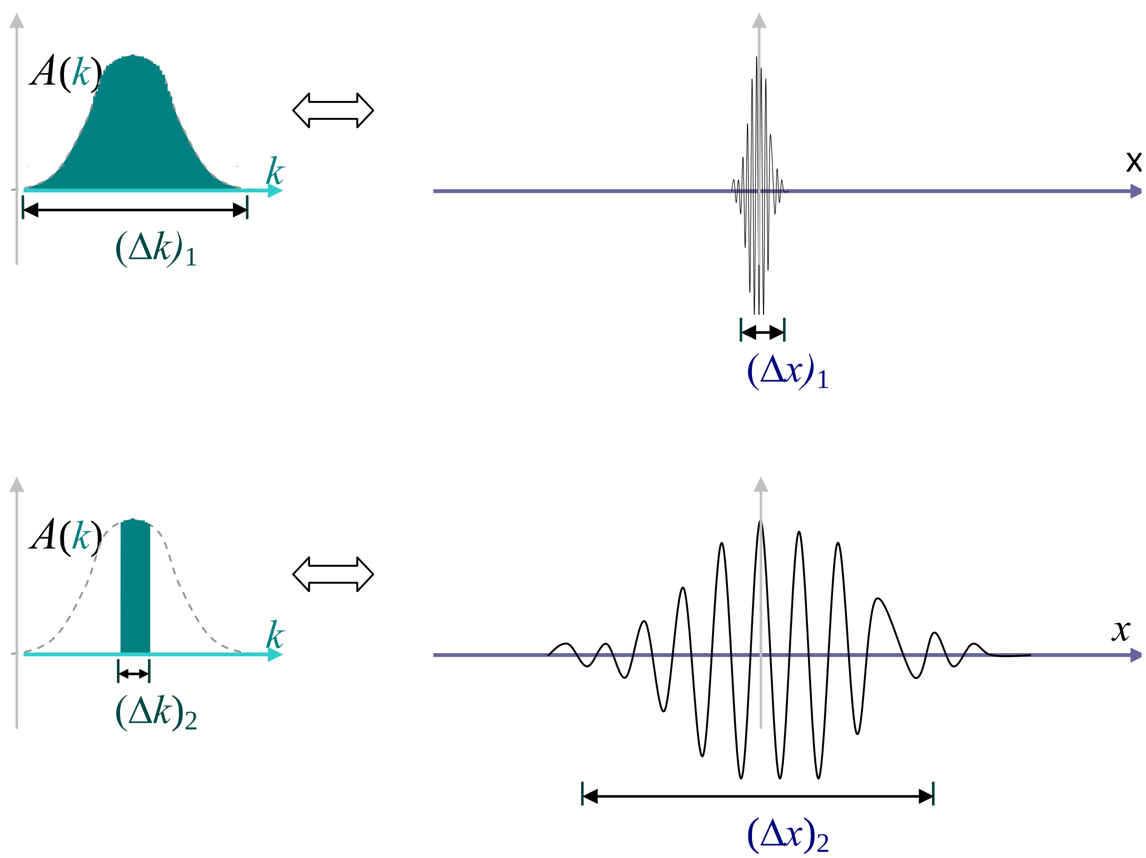
What a bout the variation of the pulse-size as the number of wavevectors (all with values within the range Δk*) increases?*

Fig. 4.15 above already suggests that the size should decrease. In effect, as the number of harmonic waves increases, the multiple addition of waves tends to average out to zero, unless x =0 or for values of x very small; that is the pulse becomes narrower.

Thus, we now can understand better the property stated in a previous paragraphs above (see expression (29) above, where the properties of the Fourier transform were being discussed.)

the more localized the function the broader its spectral response; and vice versa.

In effect, notice in the previous figure that if we were to increase the range Δk , the corresponding range Δx of values of the x coordinate for which all the harmonic wave component can approximately interfere constructively would be reduced; and vice versa.



In short:

$$\Delta x \sim \frac{1}{\Delta k} \tag{45}$$

This is a general property of the Fourier analysis of waves (in principle, it has nothing to do with Quantum Mechanics.

We will see later that one way to describe QM is within the framework of Fourier analysis. In this context, some of the mathematical terms are identified (via the de Broglie hypothesis) with the particle's physical variables, which, accordingly, become subjected to the relationship indicated in (45). The realization that physical variables are subjected to the relationship (45) constitutes one of the cornerstones of Quantum Mechanics. We will explore this concept in the following sections.