COMPLEX NUMBERS

1. **Definition of complex numbers**
   - **Complex conjugate**
   - **Magnitude**

   Operations
   - **Addition,**
   - **Multiplication,**
   - **Reciprocal number**

2. **Representation of complex numbers in polar form**

   The Euler's representation $z = a + ib = Ae^{i\theta}$

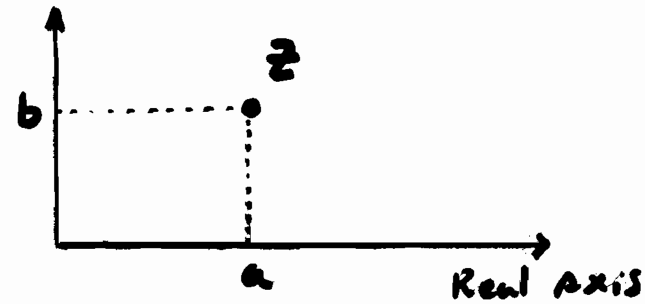
3. **Solving differential equations using complex variable**

COMPLEX NUMBERS (Section 2.5)

$z = a + ib$

where a, b are reals

$i^2 = -1$



Conjugate of $z = a - ib \equiv z^*$

Magnitude of $z = |z| = \sqrt{a^2 + b^2}$

OPERATIONS:

$z_1 = a_1 + i b_1$

$z_2 = a_2 + i b_2$

Addition

$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$

Multiplication

$z_1 z_2 = (a_1 a_2 - b_1 b_2) + i\,(a_1 b_2 + a_2 b_1)$

Reciprocal number

$$\frac{1}{z} = \frac{1}{a+ib} = \frac{1}{a+ib}\,\frac{a-ib}{a-ib} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$$

$z = a + ib$

It is called the REAL part

the COMPLEX part

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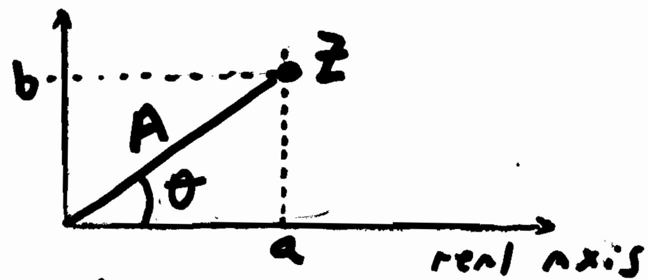


• The Euler's formula

Let $z = A(\cos\theta + i\sin\theta) = z(\theta)$

Notice

$$\frac{dz}{d\theta} = A(-\sin\theta + i\cos\theta)$$
$$= i(A\cos\theta + i\sin\theta) = iz$$

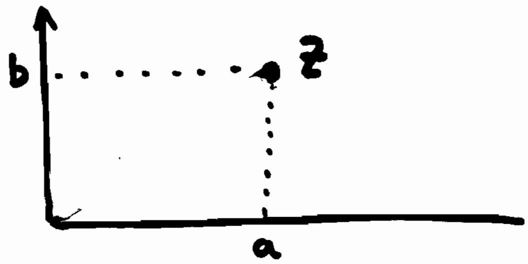
We recall that, when working with real numbers:

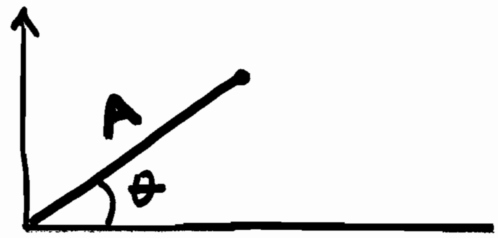
$$\frac{dAe^{\alpha\theta}}{d\theta} = \alpha(Ae^{\alpha\theta})$$

So, to the effects of derivations (and integrations) the complex variable z behaves as,

$$z = A(\cos\theta + i\sin\theta) = Ae^{i\theta}$$ EULER's formula

So, we treat these two expressions as indistinguishables





$$\boxed{z = a + i\,b}$$

$$A = (a^2 + b^2)^{1/2} \qquad \theta = \tan^{-1}\frac{b}{a}$$

$$\boxed{z = A(\cos\theta + i\sin\theta)}$$

To the effects of derivatives (and integrals) we have found,

$$\boxed{z = A e^{i\theta}}$$ Euler's formula

Further check of Euler's formula:

↳ Is it valid for $\frac{1}{z}$?

that is, it is $\frac{1}{z} = \frac{1}{A} e^{-i\theta}$?

Indeed,

Since $\frac{1}{z} = \frac{1}{a+ib} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$

then

$$\frac{1}{z} = \frac{A\cos\theta}{A^2} - i\frac{A\sin\theta}{A^2} = \frac{1}{A}\underbrace{(\cos\theta - i\sin\theta)}_{e^{-i\theta}}$$

$$= \frac{1}{A}e^{-i\theta}$$

Thus,

$$\frac{1}{z} = \frac{1}{a+ib} = \frac{1}{Ae^{i\theta}} = \frac{e^{-i\theta}}{A}$$

As demonstrated above

↳ What about $z_1 z_2$?

If $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$

Is $z_1 z_2 = A_1 A_2 e^{i(\theta_1+\theta_2)}$?

In short,

- Anytime we write $\mathbf{A}\mathbf{e}^{j\theta}$

  we actually mean **Acos(θ) + *j* A Sin(θ)**
- $\mathbf{A}\mathbf{e}^{j\theta}$ is simply easier to manipulate

Solving differential equations using complex variables

- Consider the following equation, where all the quantities are real numbers,

$$m\frac{d^2x}{dt^2}+b\frac{dx}{d}+kx=F_oCos(\omega t) \qquad (1)$$

This is the Eq. that governs the dynamic response of an oscillator under the influence of a harmonic external force $F_oCos(\omega t)$.

We are looking for a solution $x = x(t)$

- We can always consider a parallel Eq.

$$m\frac{d^2y}{dt^2}+b\frac{dy}{d}+ky=F_oSin(\omega t)$$

Notice the force is now $F_oSin(\omega t)$

(Different force, different solution; hence the use of *y* instead of *x*.)

Judiciously, and since the Eq. is linear, we multiply the Eq. by the complex number *j*; thus

$$m\frac{d^2 jy}{dt^2} + b\frac{djy}{d} + kjy = F_o jSin(\omega t) \qquad (2)$$

- Adding (1) and (2)

$$m\frac{d^2[x + jy]}{dt^2} + b\frac{d[x + jy]}{d} + k[x + jy] = F_o[Cos(\omega t) + jSin(\omega t)]$$

By defining

$$z = x + jy \qquad (3)$$

The above Eq. takes the form

$$m\frac{d^2 z}{dt^2} + b\frac{dz}{d} + kz = F_o e^{j\omega t} \qquad (4)$$

Compare Eq. (4) with Eq. (1)

- Thus, if we manage to solve Eq (4) using complex variable (that is, we find z), then the Real solution x that we were looking for would be given by taking the real part pf the complex solution z

$$x = \mathcal{Real}\,(z) \qquad (5)$$

Indeed, the above method can be solve for the following case:

$$m_e\frac{d^2 x}{dt^2} = q_e E_o Cos(\omega t) - m\gamma\frac{dx}{dt} - kx \qquad (6)$$

Following the procedure above, the complex variable version becomes,

$$m_e \frac{d^2 z}{dt^2} = q_e E_o e^{j\omega t} - m\gamma \frac{dz}{dt} - kz$$

If we manage to solve this Eq, that is, we find z=z(t), then the real part of that solution will be the solution of Eq. (6).

Since the external force is harmonic, we anticipate (we guess) the solution will be also harmonic. Thus, let's propose:

$$z = [z_o e^{j\varphi}] e^{j\omega t} = z_o \, e^{j(\omega t + \varphi)}$$

(notice the Euler's form in this expression)

$$\frac{dx}{dt} = \frac{d}{d} x_o e^{j\omega t} = x_o j\omega e^{j\omega t}$$

$$\frac{d^2 x}{dt^2} = \frac{d}{d} \frac{dx}{d} =$$ ... etc