Complex Variable, Addition of Waves by the Phasors Method

COMPLEX NUMBERS

Addition, multiplication, reciprocal number

Euler's representation $z = a + ib = Ae^{i\theta}$

REPRESENTATION of TRAVELING HARMONIC WAVES in COMPLEX VARIABLE: PHASORS

The concept of phasors

Phasors are complex numbers (they are not vectors)

Addition of (real) waves using phasors

Waves as the real components of phasors

Graphic interpretation

Adding waves of the same frequency and wavelength

Example: Addition of two waves. Calculation of magnitude and phase

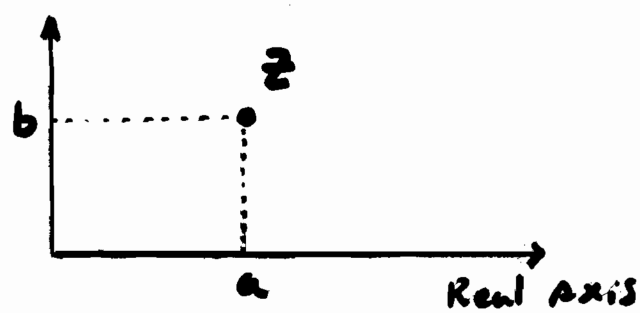
Generalization to add many waves.

COMPLEX NUMBERS (Section 2.5)

$z = a + i\,b$

where a, b are reals

$i^2 = -1$



Conjugate of $z = a - ib \equiv z^*$

Magnitude of $z = |z| = \sqrt{a^2 + b^2}$

OPERATIONS:

$z_1 = a_1 + i\,b_1$

$z_2 = a_2 + i\,b_2$

Addition

$z_1 + z_2 = (a_1 + a_2) + i\,(b_1 + b_2)$

Multiplication

$z_1 z_2 = (a_1 a_2 - b_1 b_2) + i\,(a_1 b_2 + a_2 b_1)$

Reciprocal number

$$\frac{1}{z} = \frac{1}{a+ib} = \frac{1}{a+ib}\,\frac{a-ib}{a-ib} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$$

$z = a + i\,b$

It is called the REAL part

the COMPLEX part

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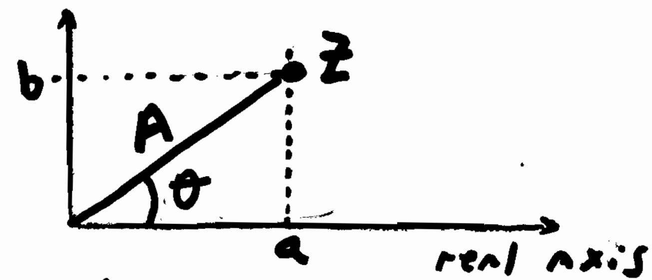


• The Euler's formula

Let $z = A(\cos\theta + i\sin\theta) = z(\theta)$

Notice

$$\frac{dz}{d\theta} = A(-\sin\theta + i\cos\theta)$$
$$= i(A\cos\theta + i\sin\theta) = iz$$

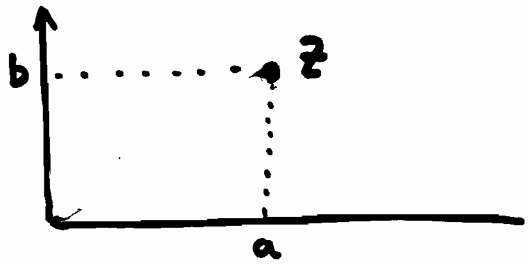
We recall that, when working with real numbers:

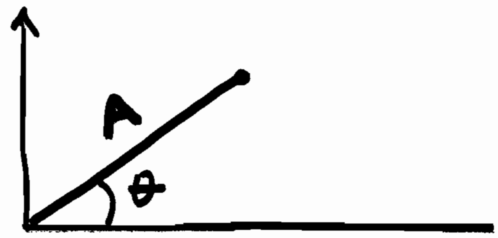
$$\frac{dAe^{\alpha\theta}}{d\theta} = \alpha(Ae^{\alpha\theta})$$

So, to the effects of derivations (and integrations) the complex variable z behaves as,

$$z = A(\cos\theta + i\sin\theta) = Ae^{i\theta}$$ EULER's formula

So, we treat these two expressions as indistinguishables





$$\boxed{z = a + i\,b}$$

$$A = (a^2 + b^2)^{1/2} \qquad \theta = \tan^{-1}\frac{b}{a}$$

$$\boxed{z = A(\cos\theta + i\sin\theta)}$$

To the effects of derivatives (and integrals) we have found,

$$\boxed{z = A e^{i\theta}}$$ Euler's formula

Further check of Euler's formula:

↳ Is it valid for $\frac{1}{z}$?

that is, it is $\frac{1}{z} = \frac{1}{A}\,e^{-i\theta}$?

Indeed,

Since $\frac{1}{z} = \frac{1}{a+ib} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$

then

$$\frac{1}{z} = \frac{A\cos\theta}{A^2} - i\frac{A\sin\theta}{A^2} = \frac{1}{A}\underbrace{(\cos\theta - i\sin\theta)}_{e^{-i\theta}}$$

$$= \frac{1}{A}e^{-i\theta}$$

Thus,

$$\frac{1}{z} = \frac{1}{a+ib} = \frac{1}{Ae^{i\theta}} = \frac{e^{-i\theta}}{A}$$

As demonstrated above

↳ What about $z_1 z_2$?

If $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$

Is $z_1 z_2 = A_1 A_2 e^{i(\theta_1+\theta_2)}$?

REPRESENTATION of a TRAVELING HARMONIC WAVE in COMPLEX VARIABLE: PHASORS

WAVE $= \Psi(x,t)$

$= A\cos(kx - wt + \alpha)$

where

$k = \frac{2\pi}{\lambda}$

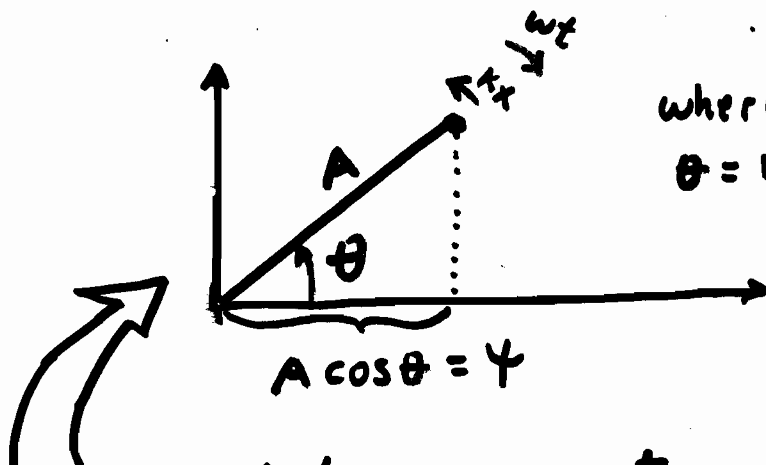
$w = \frac{2\pi}{T}$

Let's call $\theta = kx - wt + \alpha$

thus, $\boxed{\Psi(x,t) = A\cos(\theta)}$ this is a REAL measurable wave

Notice, Ψ can be interpreted as the REAL component of the associated complex variable

$$z = Ae^{i\theta}$$



where

$\theta = kx - wt + \alpha$

The rotating segment and it associated angle together constitute a PHASOR

$\Psi = \text{Real}\{z\}$

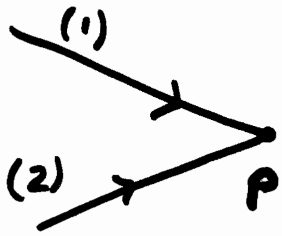
ADDITION OF WAVES using the method of PHASORS

SECTION 2.6
7.1.3

Given,

wave 1 $\Psi_1 = A_1 \cos(k_1 x - w_1 t + \alpha_1) = A_1 \cos\theta_1$

wave 2 $\Psi_2 = A_2 \cos(k_2 x - w_2 t + \alpha_2) = A_2 \cos\theta_2$



At point P the waves add up.

A convenient way to add waves is the method of phasor

$$\Psi_1 = A_1 \cos\theta_1 = \text{Real}\{\underbrace{A_1 e^{i\theta_1}}_{\text{phasor } z_1}\}$$

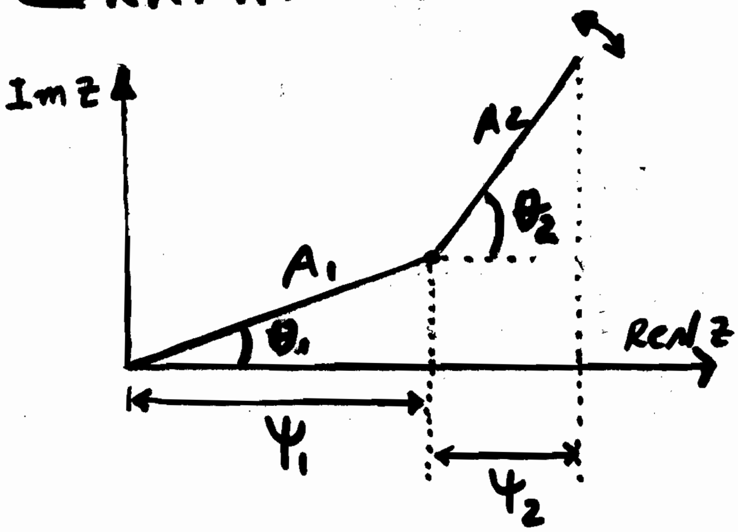
$$\Psi_2 = A_2 \cos\theta_2 = \text{Real}\{\underbrace{A_2 e^{i\theta_2}}_{\text{phasor } z_2}\}$$

$$\Psi_1 + \Psi_2 = \text{Real}\{\underbrace{A_1 e^{i\theta_1} + A_2 e^{i\theta_2}}\}$$

working with the complex variable is sometimes easier to find a total sum.

Once the total sum (a complex number) is found, its real part will be $\Psi_1 + \Psi_2$

GRAPHIC INTERPRETATION

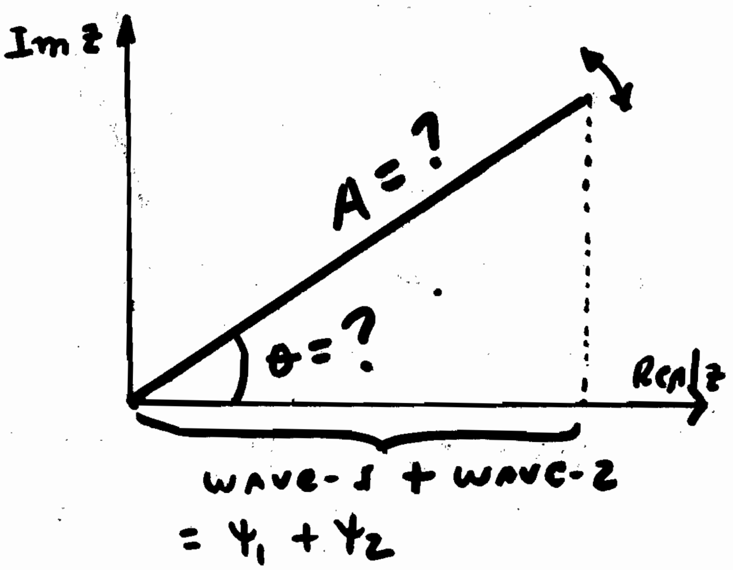


$\theta_1 = k_1 x - \omega_1 t + \alpha_1$

$\theta_2 = k_2 x - \omega_2 t + \alpha_2$

$\psi_1 = A_1 \cos\theta_1$

$\psi_2 = A_2 \cos\theta_2$



STEP 1: construct the corresponding phasors

$z_1 = A_1 e^{i\theta_1}$

$z_2 = A_2 e^{i\theta_2}$

(see top graph)

Notice also:
As x and t change, the phasors rotate

STEP 2 work in the complex world

$$z_1 + z_2 = A_1 e^{i\theta_1} + A_2 e^{i\theta_2}$$

which will have the form

$$= A e^{i\theta}$$ (see graph above)

STEP 3 $\psi_1 + \psi_2$ = Horizontal component of $\{A e^{i\theta}\}$

CASE: Adding HARMONIC WAVES of the same frequency and the same wavelength

$$\psi_1 = A_1 \cos(kx - \omega t + \alpha_1) = A_1 \cos(\theta_1)$$

$$\psi_2 = A_2 \cos(kx - \omega t + \alpha_2) = A_2 \cos(\theta_2)$$

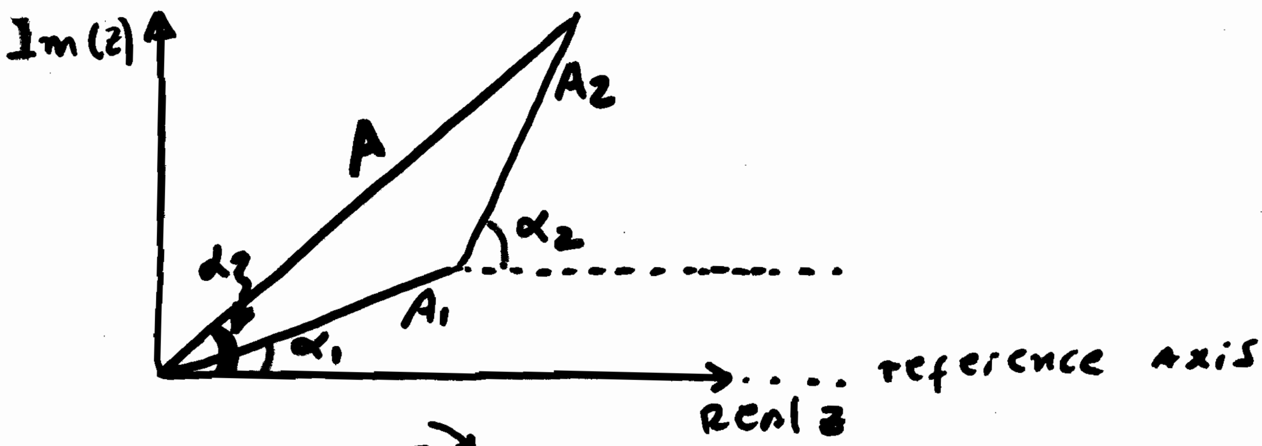
$\Rightarrow$

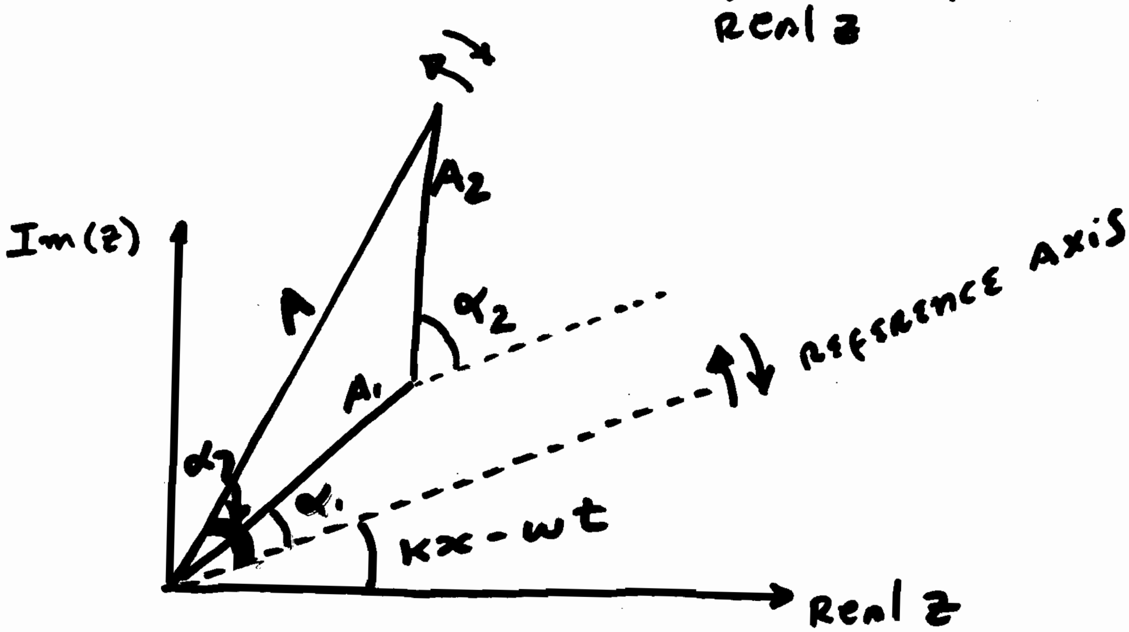
$$z_1 = A_1 e^{i\theta_1} = A_1 e^{i\alpha_1} e^{i(kx - \omega t)}$$

$$z_2 = A_2 e^{i\theta_2} = A_2 e^{i\alpha_2} e^{i(kx - \omega t)}$$

$$z_1 + z_2 = \left(A_1 e^{i\alpha_1} + A_2 e^{i\alpha_2}\right) e^{i(kx - \omega t)}$$

It should be clear from here that all the phasors, z_1, z_2 and $z_1 + z_2$ rotate synchronously (graphic interpretation).





key recipe

So, the problem of adding

$$\psi_1 = A_1 \cos(kx - wt + \alpha_1)$$

$$\psi_2 = A_2 \cos(kx - wt + \alpha_2)$$

reduces to adding

$$A_1 e^{i\alpha_1} + A_2 e^{i\alpha_2}$$

This sum will have the form

$A e^{i\alpha}$ — A and α to be determined in terms of A_1, A_2, α_1 and α_2

Finding A and α

$$A e^{i\alpha} = A_1 e^{i\alpha_1} + A_2 e^{i\alpha_2}$$

- Taking the magnitude square (Remember $|z|^2 = z z^*$, $z = Ae^{i\alpha} \Rightarrow z^* = A e^{-i\alpha}$)

$$|A e^{i\alpha}|^2 = (A_1 e^{i\alpha_1} + A_2 e^{i\alpha_2})(A_1 e^{-i\alpha_1} + A_2 e^{-i\alpha_2})$$

$$= A_1^2 + A_2^2 + A_1 A_2 \left[e^{i(\alpha_2-\alpha_1)} + e^{-i(\alpha_2-\alpha_1)} \right]$$

$$A^2 = A_1^2 + A_2^2 + 2A_1A_2\cos(\alpha_2-\alpha_1)$$ ← solution

- Using Euler's identity

$$A \sin\alpha = A_1 \sin\alpha_1 + A_2 \sin\alpha_2$$

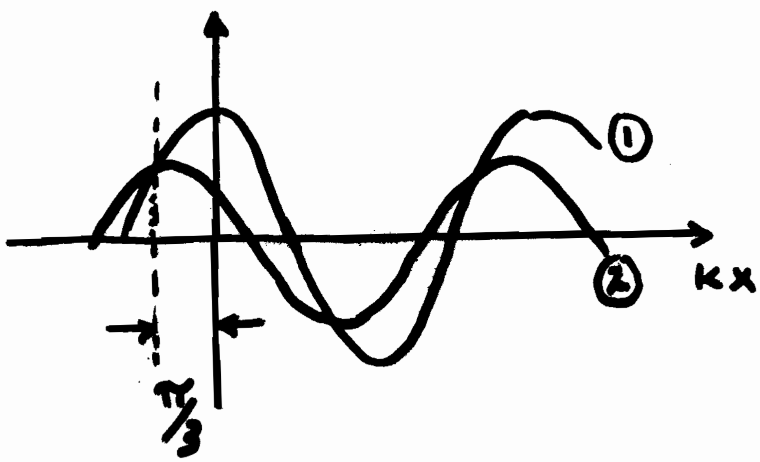
$$A \cos\alpha = A_1 \cos\alpha_1 + A_2 \cos\alpha_2$$

$$\tan\alpha = \frac{A_1 \sin\alpha_1 + A_2 \sin\alpha_2}{A_1 \cos\alpha_1 + A_2 \cos\alpha_2}$$ ← solution

EXAMPLE Two waves travel along a string in the same direction

$$\psi_1 = 4\,\text{mm} \cos(kx - \omega t)$$

$$\psi_2 = 3\,\text{mm} \cos\left(kx - \omega t + \frac{\pi}{3}\right)$$



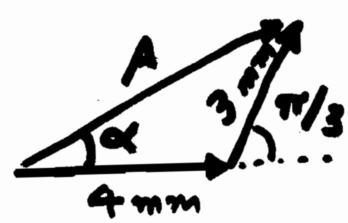
Find $\psi_1 + \psi_2$

SOLUTION

$\psi_1 \longrightarrow 4\,\text{mm}$

$\psi_2 \longrightarrow 3\,\text{mm}\, e^{i\pi/3}$

REAL — COMPLEX VARIABLE



$$4\,\text{mm} + 3\,\text{mm}\, e^{i\pi/3} = A e^{i\alpha}$$

From the phasors graph

$$A^2 = 16 + 9 + 2 \times 4 \times 3 \cos \pi/3 = 37$$

$$\Rightarrow A = 6.1\ \text{mm}$$

$$\tan\alpha = \frac{3 \sin \pi/3}{4 + 3\cos \pi/3} = 0.47$$

$$\Rightarrow \alpha = 0.44\ \text{rad}$$

Thus, we have found

$$4\,\text{mm} + 3\,\text{mm}\, e^{i\pi/3} = 6.1\,\text{mm}\, e^{i\,0.44\,\text{rad}}$$

Accordingly

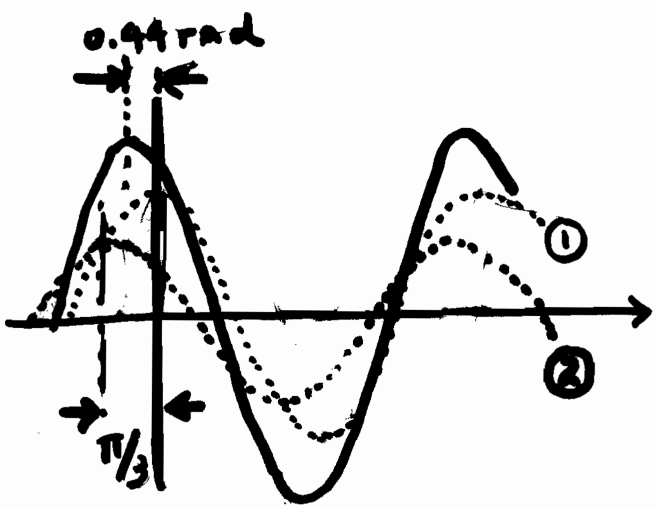
$$z_1 + z_2 = 6.1\,\text{mm}\, e^{i\,0.44\,\text{rad}}\ e^{i(kx - \omega t)}$$

$$= 6.1\,\text{mm}\, e^{i(kx - \omega t + 0.44)}$$

Back to the real world

$$\psi_1 + \psi_2 = \text{Real}\{z_1 + z_2\}$$

$$= 6.1\,\text{mm} \cos(kx - \omega t + 0.44\,\text{rad})$$

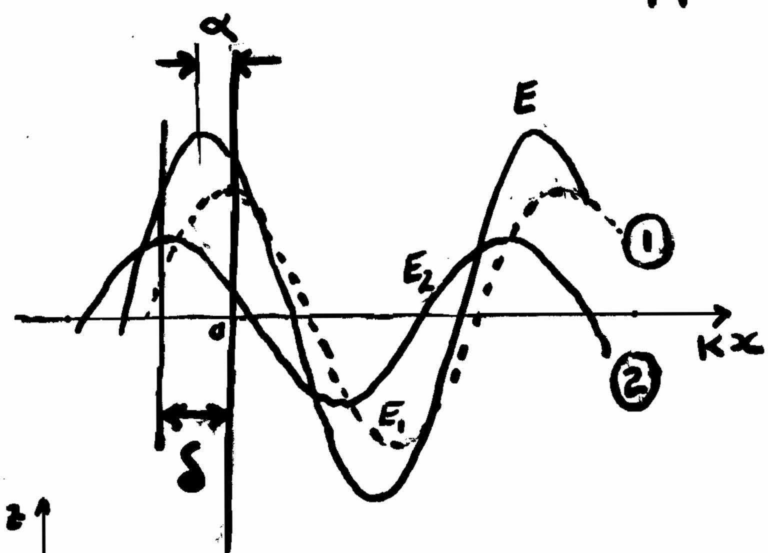


Notice:

The phase 0.44 rad of the resulting wave is between the phase values of the component waves.

$$0 < 0.44\,\text{rad} < \pi/3$$

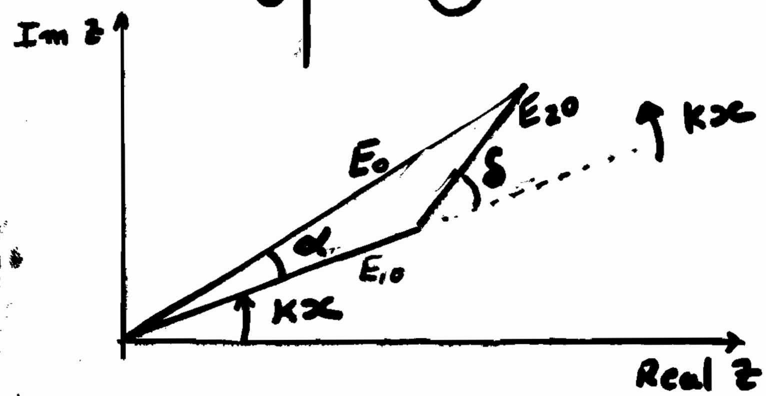
Example: Addition of two waves having a phase difference δ



$E_1 = E_{10} \cos kx$ $\quad (\alpha_1 = 0)$

$E_2 = E_{20} \cos(kx + \delta)$

δ is given



. while

$z_1 = E_{10}\, e^{ikx}$

$z_2 = E_{20}\, e^{i(kx+\delta)}$,

it is enough to consider the addition of

$E_{10} + E_{20} e^{i\delta}$ ←

. The sum will be written as $E_0 e^{i\alpha}$ ←

SOLUTION:

$$E_0^2 = E_{10}^2 + E_{20}^2 + $$
$$+ 2 E_{10} E_{20} \cos\delta$$

$$\tan\alpha = \frac{E_{20} \sin\delta}{E_{10} + E_{20}\cos\delta}$$

$$z_1 + z_2 = \left[E_0 e^{i\alpha}\right] e^{i(kx - \omega t)} = E_0 e^{i(kx - \omega t + \alpha)}$$

then

$$E_{10} \cos kx + E_{20} \cos(kx + \delta) = E_0 \cos(kx - \omega t + \alpha)$$

Phasors Method

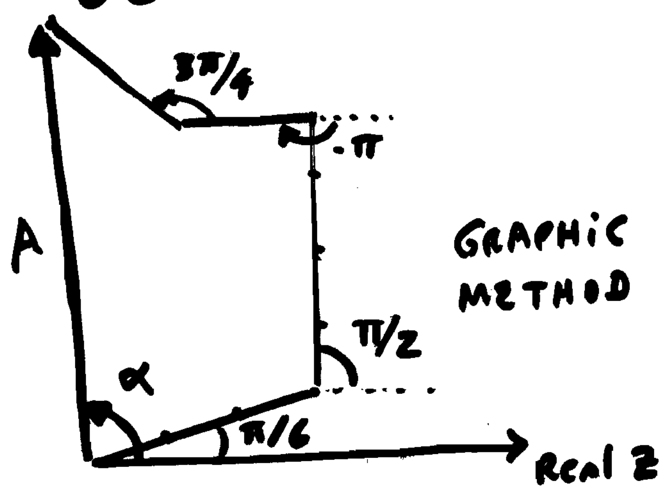
Generalization to add many waves

$$\psi = 3\cos(\omega t + \pi/6) + 4\cos(\omega t + \pi/2) + 2\cos(\omega t - \pi) + 2.5\cos(\omega t + \tfrac{3}{4}\pi)$$

Since all the waves have the same frequency, we set to add

$$3e^{i\pi/6} + 4e^{i\pi/2} + 2e^{-i\pi} + 2.5\,e^{i\frac{3}{4}\pi} =$$

$$= Ae^{i\alpha}$$

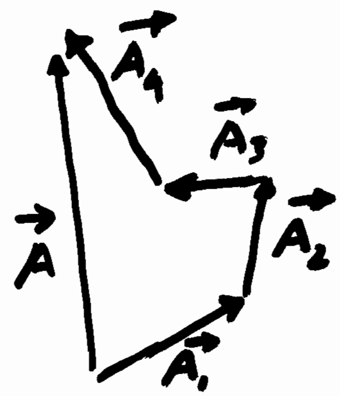


Thus

$$\psi = A\cos(\omega t + \alpha)$$

An analytical expression for A, in terms of the amplitudes A_i of the component waves, can be obtained if we consired the phasors $A_i e^{i\alpha_i}$ as if they were vectors.

$$|\vec{A}|^2 = \vec{A}\cdot\vec{A}$$
$$= \left(\sum_i \vec{A}_i\right)\cdot\left(\sum_i \vec{A}_i\right)$$



$$A^2 = \sum_i \vec{A}_i^{\,2} + \sum_i \sum_{j\neq i} \vec{A}_i\cdot\vec{A}_j$$

$$A^2 = \sum A_i^2 + \sum_i \sum_{j\neq i} A_i A_j \cos(\alpha_i - \alpha_j) \quad \leftarrow$$

and

$$\tan\alpha = \frac{\sum_i A_i \sin\alpha_i}{\sum_i A_i \cos\alpha_i} \quad \leftarrow$$