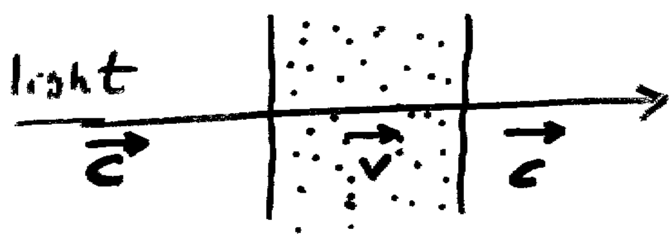
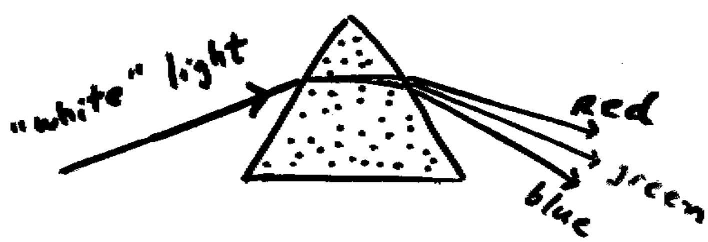
A Microscopic View of the Index of refraction

We know, the speed of light appears to slow down when it travels through a medium



$v < c$

We also know that when ordinary light (the one from the sun, for example) passes through a prism, it splits into its different color components



Since light of different colors have different frequencies, it appears then that the atoms in the prism interact differently with light of different frequencies.

On a MACROSCOPIC VIEW the two phenomena mentioned above are described by associating an index of refraction n to the medium.

For example, n is defined as $n = \frac{c}{v}$ (typically $n > 1$) and it is assumed that n depends on the frequency of the incident light; that is $n = n(\omega)$

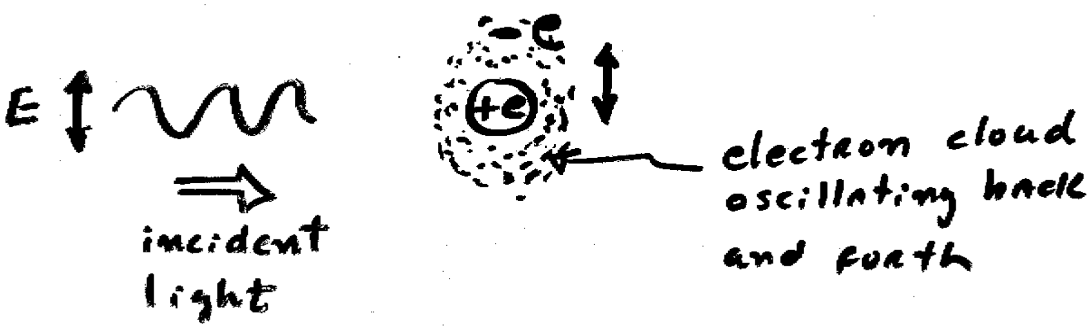
On a MICROSCOPIC VIEW, we plan to develop a mathematical model of the index of refraction n of a medium in terms of the dynamic response of the atoms (that constitute such medium) to the external excitation produced by light

How can light interact with atoms ?

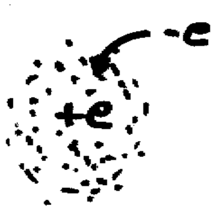
It is because
light is a traveling wave of electric fields

and
atoms contain charges

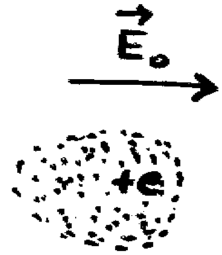
We'll see, then, that the index of refraction of a medium has its origen in the peculiar response of ~~the~~ charges (contained in the atoms) to the electric fields (from the incident light)



First, let's make a quick review on the polarization effects we learned from ELECTROSTATICS



No external electric field



An electrostatic field $\vec{E}_o$ is applied

model

$\Longleftrightarrow$

$\vec{p} = e\vec{d}$

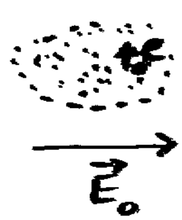
called "dipole moment"

Notice:

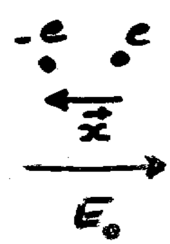
$\vec{p} \parallel \vec{E}_o$

Both point in the same direction

Alternatively, since we plan to follow the electron cloud motion, we will express $\vec{p}$ in terms of the electron mean position $\vec{x}$



$\Longleftrightarrow$



so,

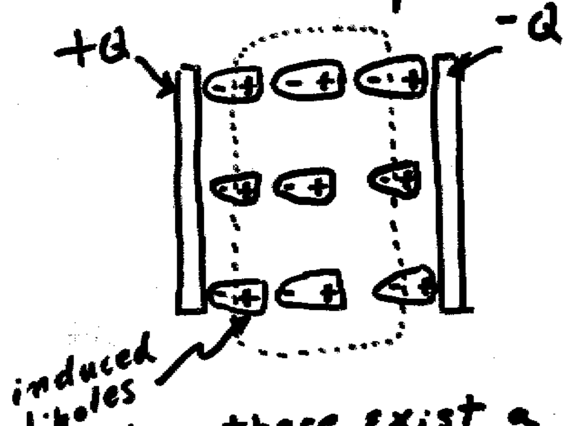
$$\boxed{\vec{p} = -e\vec{x}}$$

Notice: $\vec{p} \parallel \vec{E}_o$

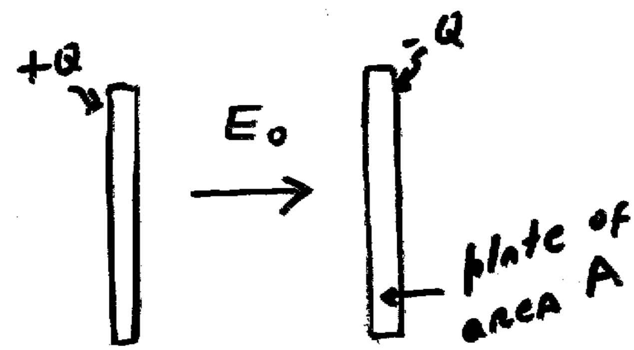
What do a collection of electric dipoles $\vec{p}$ do?

Answer: the produce collectively an additional electric field opposite to the external field $\vec{E}_o$. The net electric field, then, is smaller than E_o in magnitude.

For example:



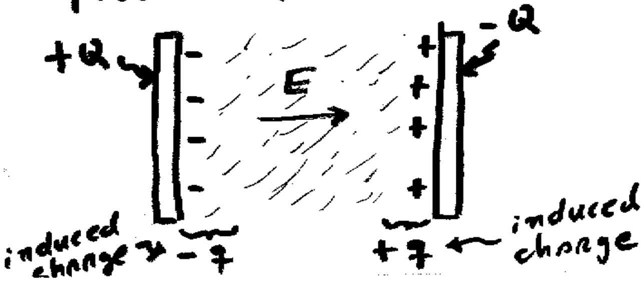
when there exist a material between the plates, it gives rise to the formation of dipoles.



when there is nothing between the plates,

$$E_o = \frac{\sigma}{\epsilon_o} = \frac{Q}{A\,\epsilon_o}$$

The above charge distribution can be oversimplified as follows:

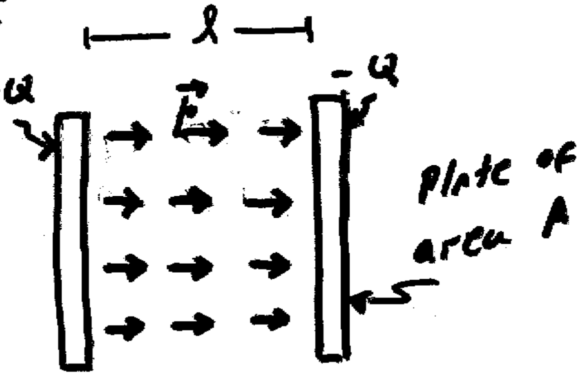


so, the net electric field is given by

$$E = \frac{Q - q}{A\,\epsilon_o}$$

How do Q and q relate ?

Let's take a unit volume and add-up all the dipoles contained in that volume; the resulting total dipole is called Polarization vector $\vec{P}$.

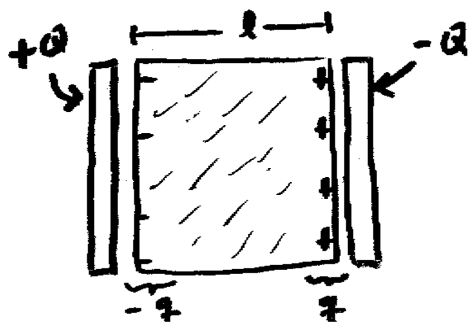


$$\vec{P} \equiv \frac{\text{dipole moment}}{\text{unit volume}}$$

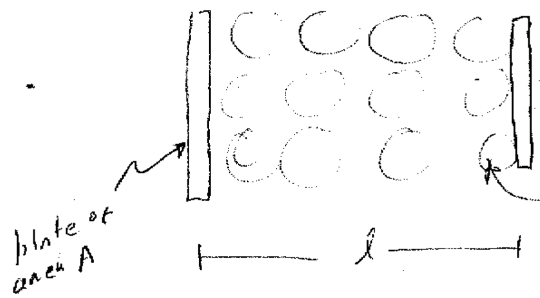
Total dipole moment contained between the plates $= P_x A l$

If we use again our oversimplified charge distribution (shown in the previous page) we obtain

Total dipole moment contained between the plates $= q\, l$



Another way to derive that $\mathbb{P} = \frac{q}{A}$ is the following:



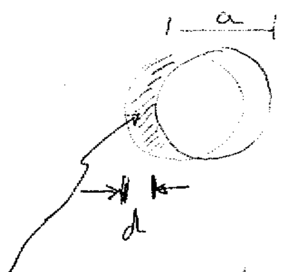
There are $\mathcal{M}$ atoms.

Each contributes with a dipole p

Then:

$$\mathrm{P} = \frac{\sum p_i}{A\,l} = \boxed{\frac{\mathcal{M}\,p}{A\,l} = \mathbb{P}} \quad (1)$$

On the other hand, in each atom, the center of the negative cloud charge does not concide with the center of the positive charge



$p = ed$

we can estimate that the amount of induced $\underbrace{\text{charge}}_{\text{per atom}}$ is $e\frac{d}{a}$

$$\boxed{q_{induced} = e\frac{d}{a}} \quad (2)$$

(Notice it make sense, because the dipole will be equal to

$q_{induced} \times a = \frac{ed}{a} \times a = ed.$)

So either we adopt the model of two charges e and $-e$ separated by a distance d

↳ or, charge $\frac{ed}{a}$ and $-\frac{ed}{a}$ separated by a distance a

In either case, the electric dipole of the atom is $p = ed$

Now $\mathcal{M}\,a^3 = A\,l$

If there are N_A atoms on the surface of the plate we have

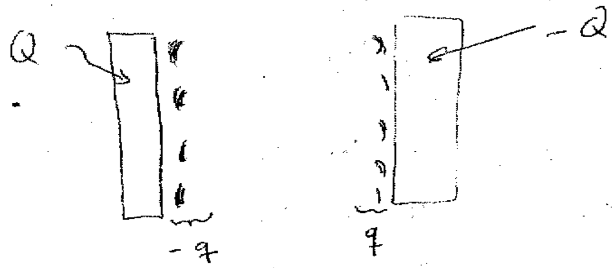
$$\mathcal{M}\,a = \frac{A}{a^2}\,l = N_A\,l \quad (3)$$

From (1)

$$\mathrm{P} = \frac{\mathcal{M}\,p}{A\,l} = \frac{N_A\,l/a \;\; p}{A\,l} = \frac{1}{A}\frac{N_A\,p}{a}$$

$$= \frac{N_A}{A}\left(\frac{ed}{a}\right) = \frac{N_A\,q_{induced}}{A} = \frac{q}{A}$$

So the picture we have is



$$q = N_A \left(q_{induced}/atom \right)$$

$$= N_A \left(e \times \frac{d}{a} \right)$$

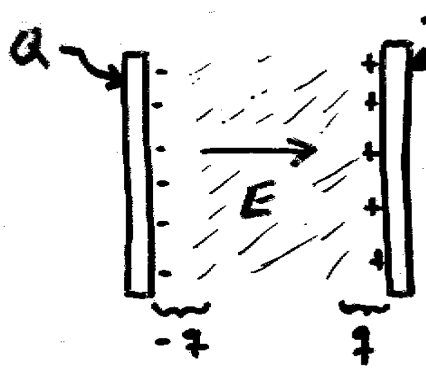
where N_A is the number of atoms in the plate.

From the last two expressions we obtain

$$\boxed{P = \frac{q}{A}}$$

The Polarization $\mathbf{P}$ of the material (total dipole moment per unit volume) is equal to the induced surface charge density

The net electric field E between the plates can then be given in terms of the polarization P



$$E = \frac{Q - q}{A\epsilon_0}$$

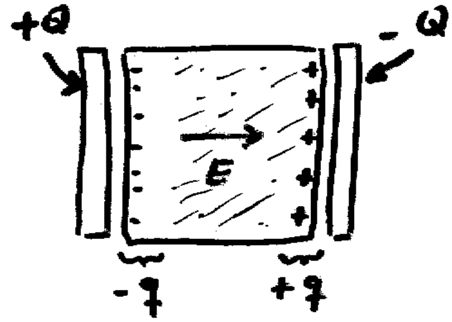
$$= \frac{Q}{A\epsilon_0} - \frac{q}{A\epsilon_0}$$

$$= \frac{Q}{A\epsilon_0} - \frac{P}{\epsilon_0}$$

$$\Rightarrow \boxed{\epsilon_0 E + P = \frac{Q}{A}}$$

we can not go any further, unless we know how P depends on E

But, still, we do not have a relationship between the free charge Q on the plates and the induced charge q on the sides of the material inside the plates.



So, we can not find the net electric field between the plates,

$$E = \frac{Q - q}{A\epsilon_0} \quad \left(\begin{matrix}\text{we do not}\\ \text{know } q \text{ yet}\end{matrix}\right)$$

All we have done so far is to identify q/A as the Polarization P of the material,

so that the equation above can, equivalently, be writen as

$$\epsilon_0 E + P = \frac{Q}{A} \quad \left(\begin{matrix}\text{we do not}\\ \text{know } P \text{ yet}\end{matrix}\right)$$

Hypothesis about the relationship between P and E

It is plausible to assume that

P will be proportional to E (E being the net electric field inside the material)

the stronger E
the bigger the separation d
the bigger the electric dipole and the bigger P

The constant of proportionnality between P and E is chosen in the following form

$$\underbrace{P}_{\text{Polarization}} = \epsilon_0 \underbrace{\chi}_{\text{called "Polarizability"}} E$$

Polarization

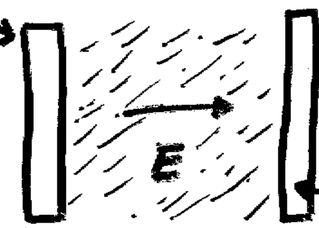
called "Polarizability"
It is determined experimentally for each material

χ is a measure of d

Replacing this relationship in the previous expression we obtain

$$\epsilon_0 E + \epsilon_0 \chi E = \frac{Q}{A} = \sigma$$

$$\Rightarrow \quad E = \frac{\sigma}{(1+\chi)\epsilon_0}$$



$$E = \frac{\sigma}{\epsilon}$$

where $\underbrace{\epsilon}_{\text{called "permitivity"}} = \epsilon_0 (1+\chi)$

called "permitivity"

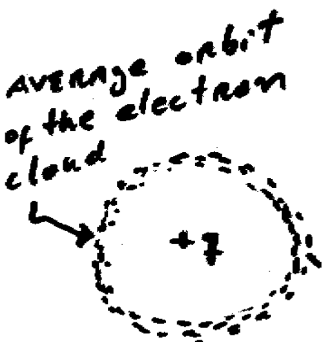
Since the electric field between the plates, when no material is present, is given by $E_0 = \frac{\sigma}{\epsilon_0}$, we realize that $E = \frac{E_0}{\epsilon/\epsilon_0} = \frac{E_0}{K}$

That is, the presence of the material reduces the electric field by a factor of $K = \epsilon/\epsilon_0$. K is called the "dielectric constant"

Materials' Responce to Alternating Electric Fields $E = E_0 \cos(\omega t)$

Electron is attracted to the nucleus by electric forces

But, because of its motion around the nucleus, a "centrifugal force" prevents the electron from getting too close the nucleus



Thus the electron ends up orbiting around the nucleus in an equilibrium orbit.

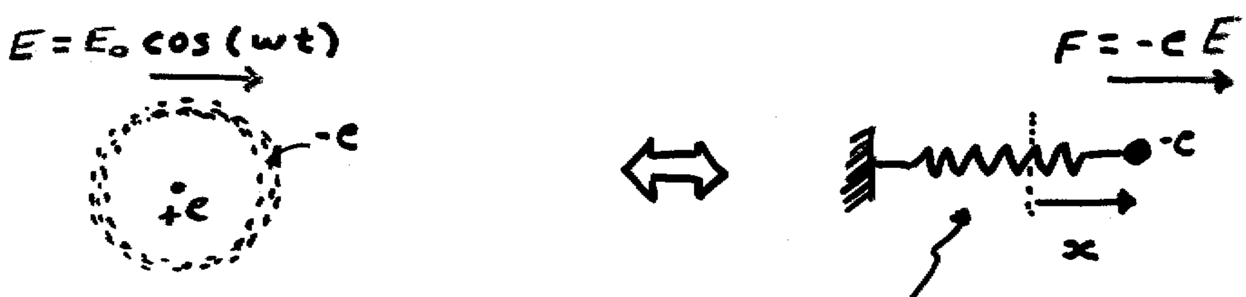
Under an external perturbation $E_0 \cos(\omega t)$ we expect the electron cloud to oscillate accordingly around its equilibrium orbit.

This view suggests we could use a mechanical model to describe the motion of the electron cloud

spring model

Since we know that atoms absorve energy very efficiently when the incident light has particular discrete frequencies, let's choose one of them, ω_0 for example. We also already know that a mechanical spring absorves energy very efficiently when perturbed by a oscillatory force of angular frequency $\sqrt{K/m}$.

Therefore, in the "spring model" of the atom we will choose the value of K as $K = m\omega_0^2$



Assume we know the atom absorves energy very efficiently when $\omega = \omega_0$

spring of spring constant $K = m\omega_0^2$

Assuming also that a viscous force $(-bv)$ is also acting on the electron, the equation of motion is

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + Kx = -eE_o\cos(\omega t)$$

or

$$m\frac{d^2(-x)}{dt^2} + b\frac{d(-x)}{dt} + K(-x) = eE_o\cos(\omega t)$$

We are already familiar with this equation. The solution is:

$$-x = A\cos(\omega t + \phi)$$

$$\text{where} \quad A = A(\omega) = \frac{eE_o/m}{\left[(\omega_o^2-\omega^2)^2 + \left(\frac{b}{m}\right)^2\omega^2\right]^{1/2}}$$

$$\phi = \phi(\omega) = \arctan\left(-\frac{(b/m)\omega}{\omega_o^2-\omega^2}\right)$$

The electric dipole of the atom will be given by

$$p = (-e)x$$

$$= \frac{e^2E_o/m}{\left[(\omega_o^2-\omega^2)^2 + (b/m)^2\omega^2\right]^{1/2}}\cos(\omega t + \phi)$$

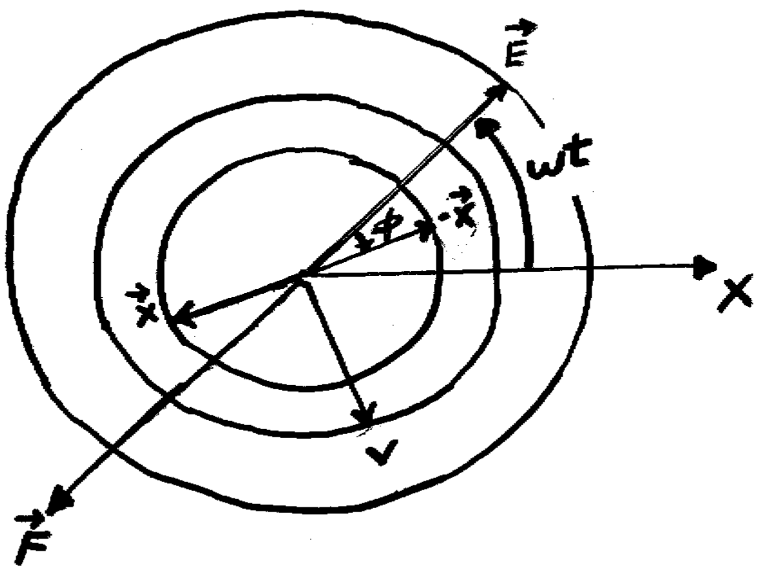
If there are N atoms per unit volume, the Polarization will be given by

$$P = N p$$

$$P(t) = \frac{N e^2 E_0 / m}{\left[(\omega_0^2 - \omega^2)^2 + (b/m)^2 \omega^2 \right]^{1/2}} \cos(\omega t + \phi)$$

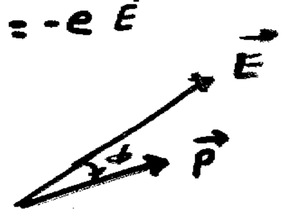
Thus, the polarization is out of phase with the excitation electric field

$$E(t) = E_0 \cos(\omega t)$$



$$\vec{p} = -e\vec{x}$$

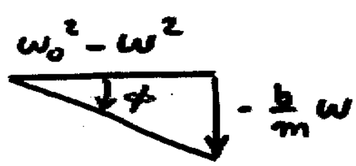
$$\vec{F} = -e\vec{E}$$



Since P and E, or equivalently x and F, are not in phase, we expect an absorption of energy by the atom from the external field

$$X = -A\cos(\omega t + \phi)$$

$$= -\left[A\cos(\phi)\cos(\omega t) - A\sin(\phi)\sin(\omega t)\right]$$



$$\cos(\phi) = \frac{\omega_0^2 - \omega^2}{\left[(\omega_0^2 - \omega^2)^2 + \left(\frac{b}{m}\right)^2\omega^2\right]^{1/2}}$$

$$\sin(\phi) = \frac{-\frac{b}{m}\omega}{\left[(\omega_0^2 - \omega^2)^2 + \left(\frac{b}{m}\right)^2\omega^2\right]^{1/2}}$$

Using the expression for $A = A(\omega)$ we obtain

$$\boxed{\frac{P}{-Ne} = X = -\frac{eE_0}{m}\frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \left(\frac{b}{m}\right)^2\omega^2}\cos(\omega t) + -\frac{eE_0}{m}\frac{(b/m)\,\omega}{(\omega_0^2 - \omega^2)^2 + \left(\frac{b}{m}\right)^2\omega^2}\sin(\omega t)} \quad (1)$$

Interpretation :

The first term represents an oscillation of the polarization,

- in phase with the external electric field at low frequencies ($\omega < \omega_0$),
- out of phase 180° at high frequencies ($\omega_0 < \omega$)

This first term does not contribute to average power absorved by the oscillator from the external electric field. (That is because when we calculate V_x this term becomes $\sin(\omega t)$ and when averaged with $E_0 \cos(\omega t)$ it will give $\langle \sin(\omega t) \cos(\omega t) \rangle = 0$)

For the reason described in the previous paragraph, the second term (the one with $\sin \omega t$) is the responsible for the energy absorption by the oscillator.

$$\boxed{P = \frac{Ne^2 E_0}{m} \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \left(\frac{b}{m}\right)^2 \omega^2} E_0 \cos(\omega t) + \frac{Ne^2 E_0}{m} \frac{(b/m)\omega}{(\omega_0^2 - \omega^2)^2 + \left(\frac{b}{m}\right)^2 \omega^2} E_0 \sin(\omega t)} \quad (2)$$

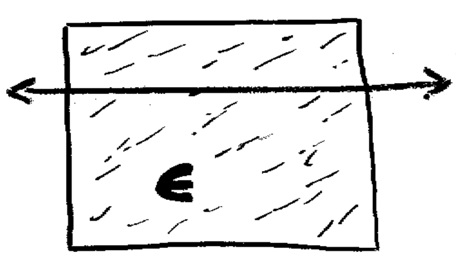
Since the first term is proportional to the applied external field (both vary a Cos(wt) with time), we will use the analogy of $P = \epsilon_0 \chi E$ used in electrostatic to define the frequency dependent polarizability as

$$\chi(w) = \frac{Ne^2}{\epsilon_0 m} \frac{w_0^2 - w^2}{\left[(w_0^2 - w^2)^2 + \left(\frac{b}{m}\right)^2 w^2\right]}$$

Accordingly, the electrical permitivity will be given by

$$\epsilon(w) = \epsilon_0 (1 + \chi)$$

$$\boxed{\epsilon(w) = \epsilon_0 + \frac{Ne^2}{m} \frac{w_0^2 - w^2}{\left[(w_0^2 - w^2)^2 + \left(\frac{b}{m}\right)^2 w^2\right]}} \quad (3)$$



$E = E_0 \cos(wt)$

On the other hand, the power per unit volume absorbed by the material from the applied external electric field is given by

$$P_{IN}(t) = N \langle F(t)\, v(t) \rangle$$

$$= N \langle \underbrace{-e E_0 \cos(\omega t)} \; \underbrace{\frac{dx}{dt}} \rangle$$

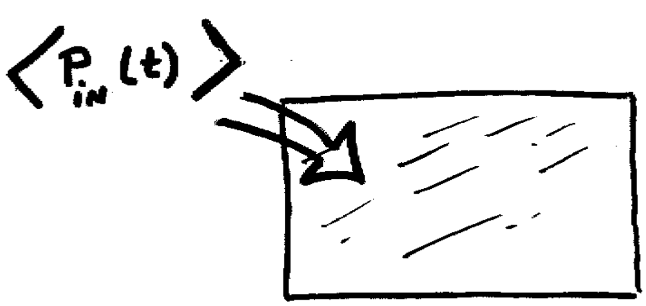
using expression 1 and the fact that

$\langle \cos(\omega t) \sin(\omega t) \rangle = 0$ and

$\langle \cos(\omega t) \cos(\omega t) \rangle = \frac{1}{2}$

we obtain

$$\boxed{\langle P_{IN}(t) \rangle = \frac{N e^2 E_0^2}{2m} \; \frac{(b/m)\,\omega^2}{(\omega_0^2 - \omega^2)^2 + \left(\frac{b}{m}\right)^2 \omega^2}} \qquad (4)$$



N: # of dipoles per unit volume

$\langle P_{IN}(t) \rangle$: average power per unit volume absorbed by the material

What does all this Polarization and forced oscillations stuff has to do with the index of refraction n of a material?

As we know, the index of refraction is defined as

$$n \equiv \frac{\text{speed of light in vacuum}}{\text{speed of light in a medium}} = \frac{c}{v}$$

Why is $c \neq v$?

The answer lies in the effects that the polarization (a bunch of electric dipoles) has on light propagation. (remember, light is a wave of electric fields)

We could argue: I can undestand that the electric fields $\vec{E}$ that constitute light will affect the material polarization $\vec{P}$. But, how is that P is going to affect E? I will answer that question this way:

In any region in space where there does exists electric and magnetic fields, the net or total field (the one that results from the light, plus the one from the dipoles that constitute the polarization of the material, plus contributions from any other source) must satisfy the Maxwell Equations (ME).

Thus, the E_1 from the light and the E_2 from the dipoles in the material must "accommodate" themselves in such a way that the net field E from these two sources complies with the law, I mean E must satisfy the ME.

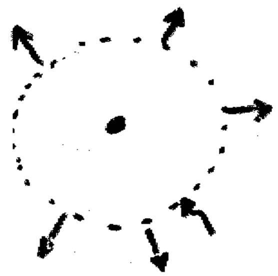
the ME equations will tell us, then, how the Polarization affects the speed of light in a material and, consequently, will provide an explanation for the index of refraction

What are the Maxwell Equations?

It is a set of 4 equations that govern the behavior of electric and magnetic fields.

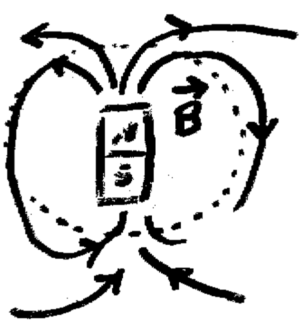
[illegible] already very familiar with them.

MAXWELL EQUATIONS



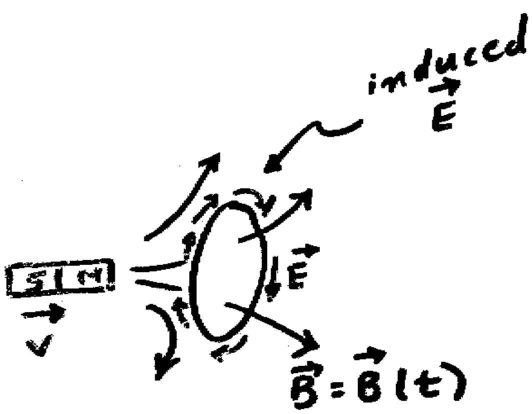
$$\int \vec{E} \cdot d\vec{A} = \frac{Q_{inside}}{\epsilon_0}$$

Gauss' law



$$\int \vec{B} \cdot d\vec{A} = 0$$

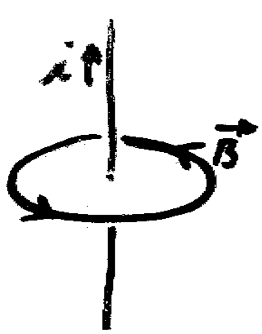
No magnetic monopole



$$\mathcal{E} = -\frac{d\Phi_M}{dt}$$

$$\int_{loop} \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_{surface} \vec{B} \cdot d\vec{A}$$

Faraday's Law



$$\int_{loop} \vec{B} \cdot d\vec{l} = \mu_0 i$$

Ampere's Law (Maxwell later modified this equation)

We will review the Maxwell Equations in more detail in the coming lectures. In the meantime let's accept the argument that

a) when the 4 equations are combined (replacing the $\vec{E}$ from the 3rd equation into the forth one, and so on) an equation results for each component E_x, E_y or E_z, which is

$$\frac{\partial^2 E}{\partial x^2} - \frac{1}{\left(\frac{1}{\epsilon_0 \mu_0}\right)} \frac{\partial^2 E}{\partial t^2} = 0$$

this is nothing but the wave equation

$$\frac{\partial^2 E}{\partial x^2} - \frac{1}{C^2} \frac{\partial^2 E}{\partial t^2} = 0$$

E is a wave that travels with velocity c. which allows us to identify c with

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$$

Using the tabulated values for ϵ_0 and μ_0 one obtains

$c = 300{,}000$ km/s

which is numerically equal to the speed

Incidentally, the fact that $\frac{1}{\sqrt{\epsilon_0 \mu_0}}$ turned out to be equal to the speed of light prompted Maxwell to believe that light was an electro-magnetic wave.
He was right.

b) When the 4 Maxwell Equations are combined taking into account that dipoles may be induced in the material (and whose oscillatory motion will produce local currents) the following equation for the net electric field results

$$\frac{\partial^2 E}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \frac{1}{\epsilon_0 c^2}\frac{\partial^2 P}{\partial t^2}$$

P: Polarization

It appears then that the electric field is not traveling any more with speed c, because of the perturbation caused by P.

Let's use

$$P = \epsilon_0 \chi E$$

where $$\chi = \frac{Ne^2}{\epsilon_0 m} \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + (\frac{b}{m})^2 \omega^2}$$

$\Rightarrow$

$$\frac{\partial^2 E}{\partial x^2} - \frac{1}{c^2}(1 + \chi)\frac{\partial^2 E}{\partial t^2} = 0$$

or

$$\frac{\partial^2 E}{\partial x^2} - \frac{1}{c^2/(1+\chi)} \frac{\partial^2 E}{\partial t^2} = 0$$

this equation describes a wave traveling with speed V

$$V = \frac{c}{\sqrt{1+\chi}}$$

Since $1+\chi = \frac{\epsilon}{\epsilon_0}$, then

$$V = \frac{c}{\sqrt{\epsilon/\epsilon_0}}$$

the index of refraction is then given by

$$n = \sqrt{\frac{\epsilon}{\epsilon_0}}$$

where

$$\epsilon = \epsilon(\omega) = \epsilon_0 + \frac{Ne^2}{m} \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + (\frac{b}{m})^2 \omega^2}$$

or, simply

$$n^2 = 1 + \frac{Nc^2}{\epsilon_0 m} \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + (\frac{b}{m})^2 \omega^2}$$