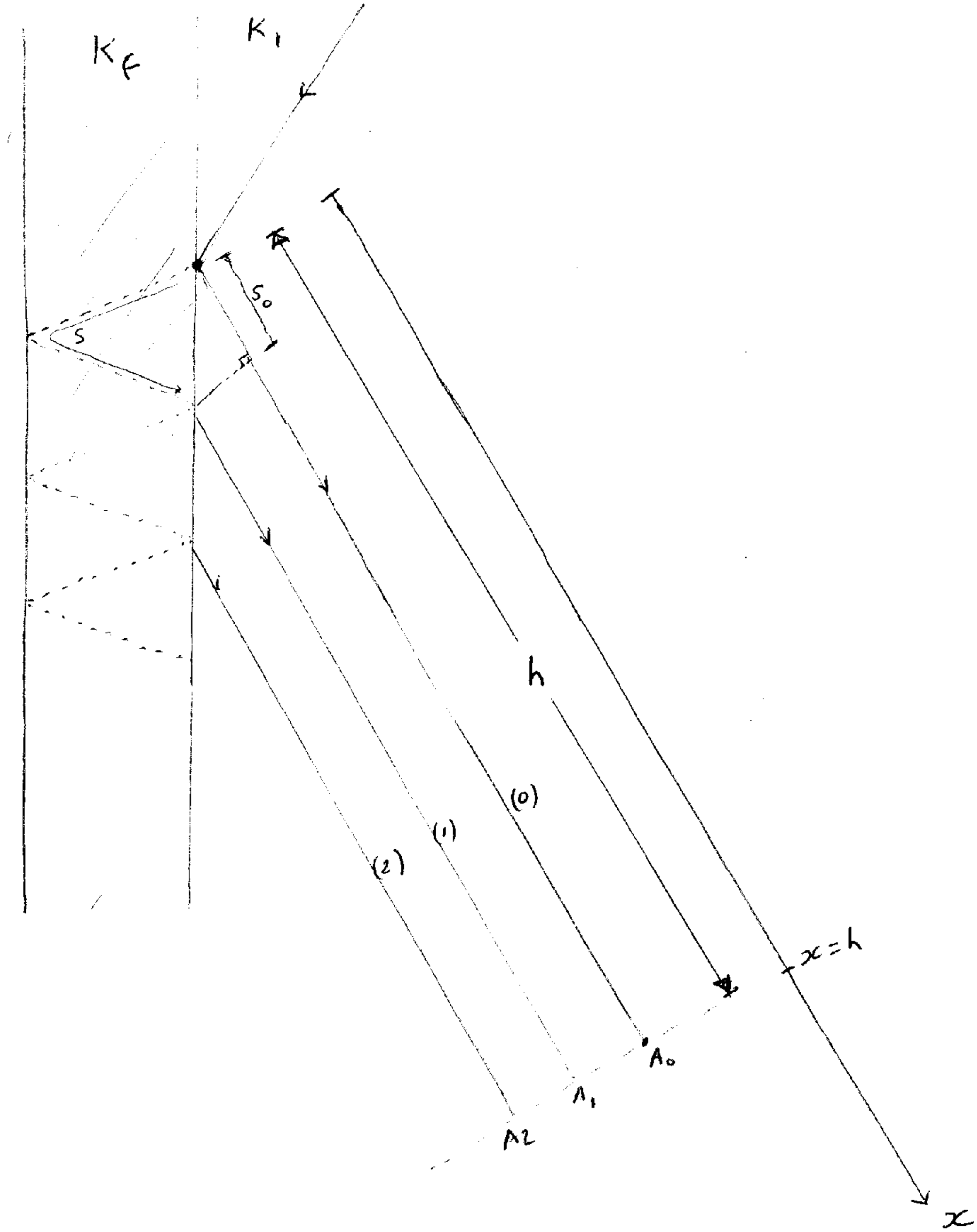
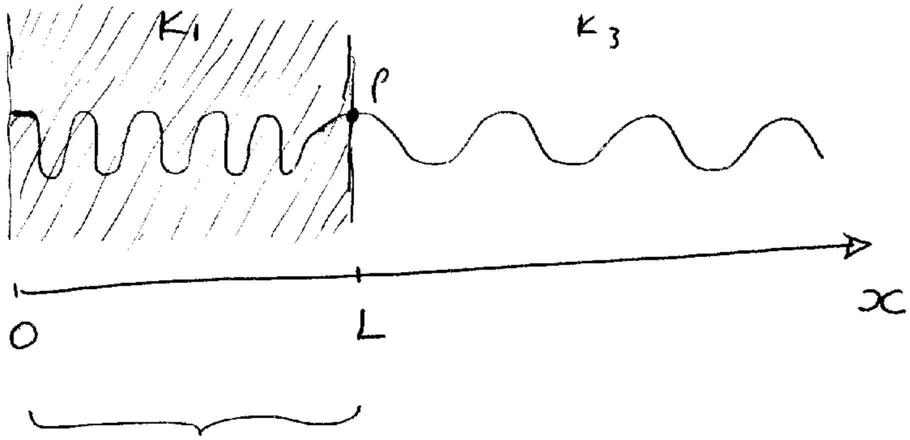
This lecture will not be included in the final exam

MULTIPLE-BEAM INTERFERENCE



We'll be using the following form to described a wave that has undergone a phase shift



phase at P : $K_1 L$

$$\cos\left[K_3(x-L)-\omega t + K_1 L\right]$$

Notice this description reflects the fact that at $x = L$ the phase of the wave is $K_1 L$

- Accumulated phase of ray (o) at A_o (i.e. $x = h$):

$$\Lambda_o = k_1 h$$

For $x > h$, the ray (o) can be expressed as

$$e^{i\left(k_1(x-h) - \omega t + \Lambda_o\right]}$$

which verifies that at $x = h$ the phase is $-\omega t + \Lambda_o$

$$= e^{i[k_1(x-h) - \omega t + k_1 h]}$$

$$= e^{i(k_1 x - \omega t)} \qquad (1)$$

- Accumulated phase of ray (1) at A_1 is

$$\Lambda_1 = k_f S + k_1 (h - S_o)$$

$$= (k_f S - k_1 S_o) + k_1 h$$

$$\Lambda_1 = \delta + k_1 h$$

where we have defined $\boxed{\delta \equiv k_f S - k_1 S_o}$

For $x > h$, ray (1) can be expressed as

$$e^{i\left(k_1(x-h) - \omega t + \Lambda_1\right)}$$

$$= e^{i\delta} e^{i(k_1 x - \omega t)} \qquad (2)$$

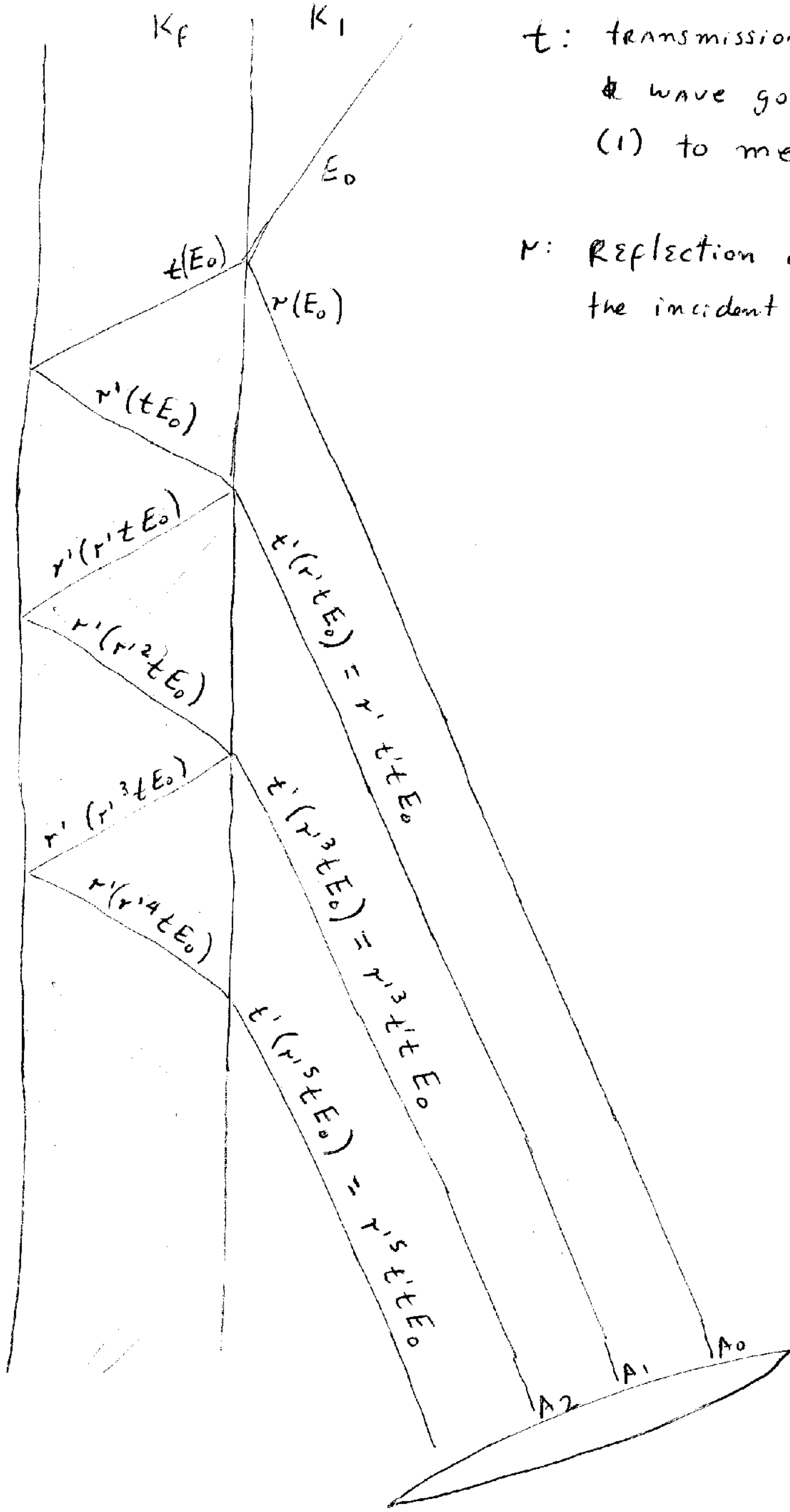
- Accumulated phase of ray (2) at A_2 is

$$\Lambda_2 = k_f 2S + k_1(h - 2S_o) = 2\delta + k_1 h$$

For $x > h$, ray (2) can be expressed as

$$e^{i[k_1(x-h) \ldots + \Lambda_2]}$$

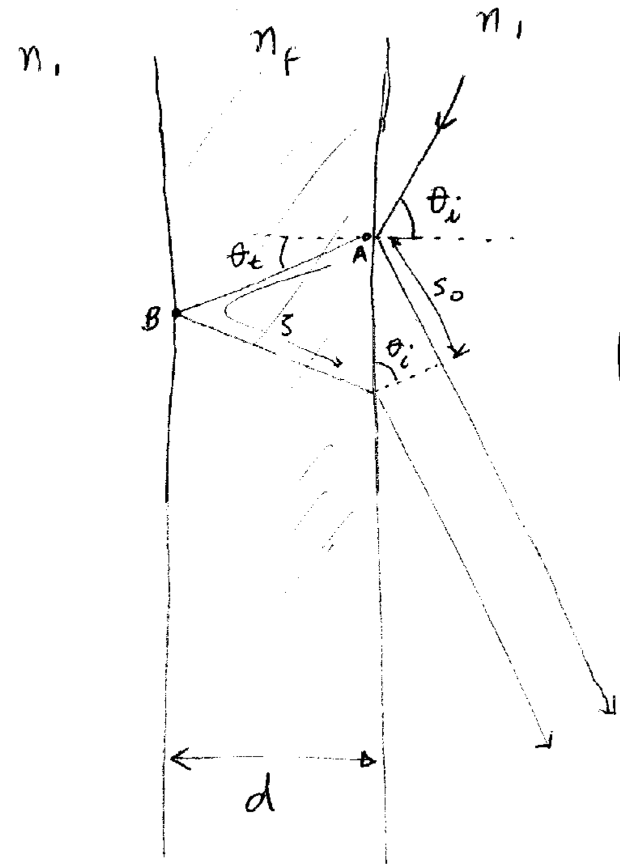
$$= e^{i2\delta} e^{i(k_1 x - \omega t)} \qquad (3)$$



t: transmission coefficient when ~~a~~ wave goes from ~~K_1 to~~ medium (1) to medium (f)

r: Reflection coefficient when the incident wave is in medium (1)

(4)



$$\delta = k_f s - k_1 s_0$$

$$= k_f\, 2AB - k_1 \underbrace{(2AB \sin\theta_t) \sin\theta_i}_{s_0}$$

$$k_f = \frac{2\pi}{\lambda_f} = \frac{2\pi}{\lambda_0} n_f$$

$$k_1 = \frac{2\pi}{\lambda_0} n_1$$

$$\delta = \frac{2\pi}{\lambda_0} 2AB \left(n_f - n_1 \sin\theta_t \sin\theta_i\right)$$

$$n_1 \sin\theta_i = n_f \sin\theta_t$$

$$\delta = \frac{2\pi}{\lambda_0} 2AB \left(n_f - n_f \sin^2\theta_t\right)$$

$$\delta = \frac{2\pi}{\lambda_0} 2AB\, n_f \cos^2\theta_t$$

$$AB \cos\theta_t = d$$

$$\delta = \frac{2\pi}{\lambda_0} 2d\, n_f \cos\theta_t$$

$$\boxed{\begin{aligned} \delta &= 2 k_f d \cos\theta_t \\ &= 2 \frac{\omega}{c} n_f d \cos\theta_t \end{aligned}}$$

Combining the results in (1), (2), (3) and (4), the total Electric field that results from the interference is given by

$$E = r E_0 e^{i(k_1 x - wt)} +$$
$$r'(t't)E_0 e^{i\delta} e^{i(k_1 x - wt)} +$$
$$r'^3(t't)E_0 e^{i2\delta} e^{i(k_1 x - wt)} +$$
$$+ \ldots$$

$$= r E_0 e^{i(k_1 x - wt)} +$$
$$\frac{1}{r'} t't E_0 \left(r'^2 e^{i\delta}\right) e^{i(k_1 x - wt)} +$$
$$\frac{1}{r'} t't E_0 \left(r'^2 e^{i\delta}\right)^2 e^{i(k_1 x - wt)} +$$
$$\frac{1}{r'} t't E_0 \left(r'^2 e^{i\delta}\right)^3 e^{i(k_1 x - wt)} +$$

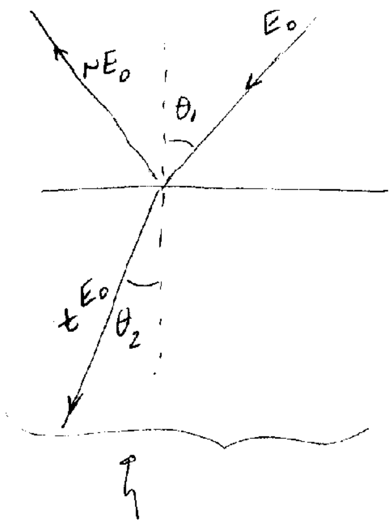
$$\ldots$$

$$= r + \frac{t't}{r'} \Big[\underbrace{\left(r'^2 e^{i\delta}\right) + \left(r'^2 e^{i\delta}\right)^2 + (\quad)^3 + \ldots}_{(\alpha + \alpha^2 + \alpha^3 + \ldots)} \Big] E_0 e^{i(k_1 x - wt)}$$

$$\left\{ \begin{aligned} (\alpha + \alpha^2 + \alpha^3 + \ldots)(1-\alpha) &= \alpha + \alpha^2 + \alpha^3 + \ldots \\ &\quad\;\; - \alpha^2 - \alpha^3 - \ldots \\ &= \alpha \\ (\alpha + \alpha^2 + \alpha^3 + \ldots) &= \frac{\alpha}{1-\alpha} \end{aligned} \right\}$$

$$= \left\{ r + \frac{t't}{r'} \frac{r'^2 e^{i\delta}}{\left[1 - r'^2 e^{i\delta}\right]} \right\} E_0 e^{i(k_1 x - wt)}$$

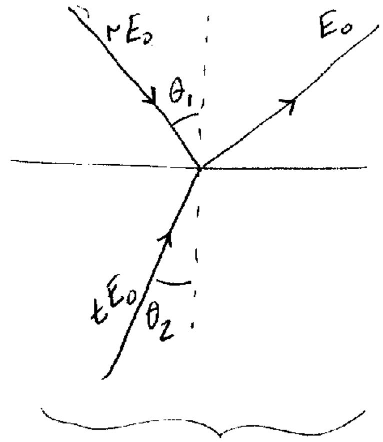
there is an interesting way to relate r, r', t and t'



If this is true

symmetry considerations indicate that

$\Rightarrow$



this situation should happen

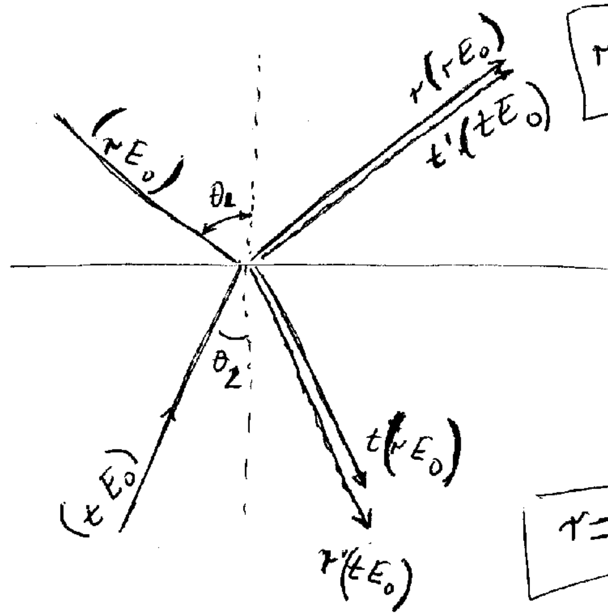
Which means:

$= r(\theta_1)$

$t = t(\theta_1)$

$r' = r'(\theta_2)$

$t' = t'(\theta_2)$



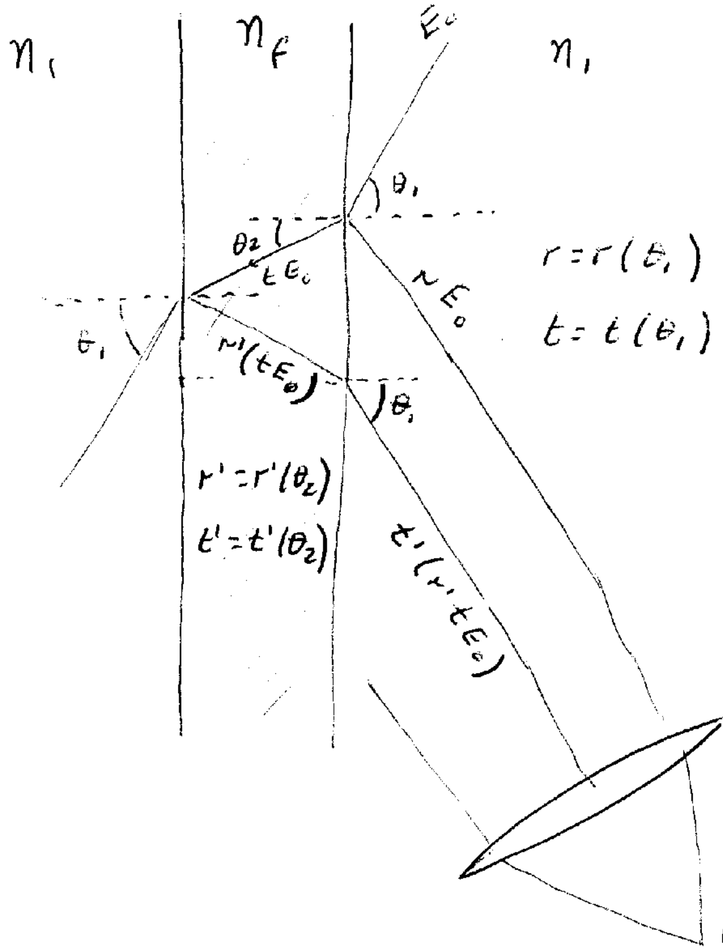
$$\boxed{r^2 + tt' = 1}$$

$$\boxed{r = r'}$$

$$r^2(\theta_1) + t(\theta_1)\,t'(\theta_2) = 1$$

$$r(\theta_1) = -r'(\theta_2)$$

Ref: Eugene Hecht, 4th Ed, p. 136



$$E = \left\{ r + \frac{t't}{r'} \frac{r'^2 e^{i\delta}}{1 - r'^2 e^{i\delta}} \right\} E_0 e^{i(k_1 x - \omega t)}$$

$$E_{refl} = \left\{ r + (1 - r^2)(-r) \frac{e^{i\delta}}{1 - r^2 e^{i\delta}} \right\} E_0 e^{i(k_1 x - \omega t)}$$

$$= r \left\{ 1 - \frac{(1 - r^2) e^{i\delta}}{1 - r^2 e^{i\delta}} \right\} E_0 e^{i(\,)} = r \left\{ \frac{1 - \cancel{r^2 e^{i\delta}} - e^{i\delta} + \cancel{r^2 e^{i\delta}}}{1 - r^2 e^{i\delta}} \right\} E_0 e^{i(k_1 x - \omega t)}$$

$$\boxed{E_{refl} = r \left\{ \frac{1 - e^{+i\delta}}{1 - r^2 e^{i\delta}} \right\} E_0 e^{i(k_1 x - \omega t)}}$$

Reflected wave

where $\delta = 2 k_f d \cos\theta_2$

$$(AB)^* = A^* B^* \quad , \quad (A+B)^* = A^* + B^*$$

$$E_{ref} \cdot E_{ref}^* = E_0^2 r^2 \frac{1 - e^{i\delta}}{1 - r^2 e^{i\delta}} \cdot \frac{1 - e^{-i\delta}}{1 - r^2 e^{-i\delta}}$$

$$= E_0^2 r^2 \frac{1 - e^{-i\delta} - e^{i\delta} + 1}{1 - r^2 e^{-i\delta} - r^2 e^{i\delta} + r^4} = E_0^2 r^2 \frac{2 - 2\cos\delta}{(1+r^4) - r^2\, 2\cos\delta}$$

$$= E_0^2 r^2 \frac{2(1-\cos\delta)}{\underbrace{(1-r^2)^2 + 2r^2}_{1+r^4} - r^2 2\cos\delta} =$$

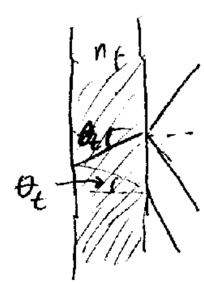
$$\boxed{|E_{refl}|^2 = E_0^2 r^2 \frac{2(1-\cos\delta)}{(1-r^2)^2 + 2r^2(1-\cos\delta)}}$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\cos 2a = \cos^2 a - \sin^2 a = 1 - 2\sin^2 a$$

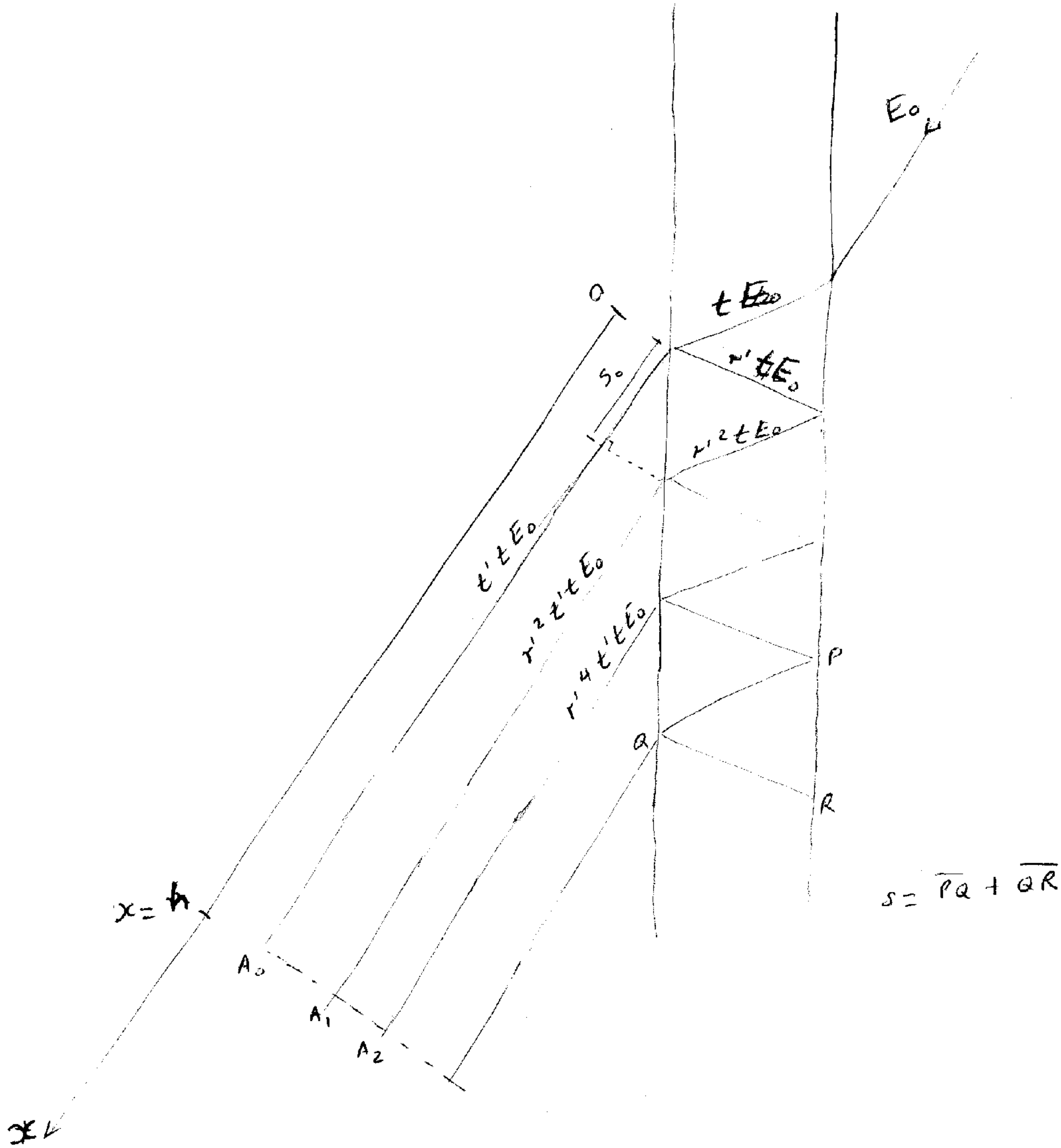
$$2\sin^2 a = 1 - \cos 2a$$

$$\boxed{|E_{refl}|^2 = E_0^2 \frac{4r^2 \sin^2\left(\frac{\delta}{2}\right)}{(1-r^2)^2 + 4r^2\sin^2\left(\frac{\delta}{2}\right)}}$$



$$\delta = 2 K_f d \cos\theta_t$$

$$= E_0^2 \frac{F \sin^2\left(\frac{\delta}{2}\right)}{1 + F\sin^2\left(\frac{\delta}{2}\right)} \qquad F \equiv \frac{2r}{1-r^2}$$



E_0
tE_0
$r'tE_0$
r'^2tE_0
O
s_0
$t'tE_0$
$r'^2t'tE_0$
$r'^4t'tE_0$
P
Q
R
$x = h$
A_0
A_1
A_2
x
$s = \overline{PQ} + \overline{QR}$

Accumulated phases (measured from the point N) | wave component (for $x>h$)

At A_0: $\Lambda_0 = k_f \frac{s}{2} + k_1 h$ | $e^{i[k_1(x-h) - \omega t + \Lambda_0]} = e^{i(k_1 x - \omega t)} e^{i k_f \frac{s}{2}}$

A_1: $\Lambda_1 = k_f \frac{3}{2} s + k_1 (h - s_0)$ | $e^{i[k_1(x-h) - \omega t + \Lambda_1]}$

$= k_f \frac{s}{2} + (k_f s - k_1 s_0) + k_1 h$ | $= e^{i(k_1 x - \omega t)} e^{i k_f \frac{s}{2}} e^{i\delta}$

$= k_f \frac{s}{2} + \delta + k_1 h$

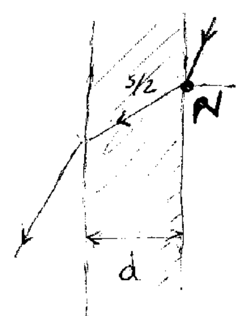
A_2: $\Lambda_2 = k_f \frac{s}{2} + k_f 2s + k_1 (h - 2s_0)$ | $e^{i(k_1 x - \omega t)} e^{i k_f \frac{s}{2}} e^{i2\delta}$

$= k_f \frac{s}{2} + 2\delta + k_1 h$

. . .

$$E = t t' E_0 \overbrace{e^{i(k_1 x - \omega t)}}^{e^{i k_f \frac{s}{2}}} \left\{ 1 + r'^2 e^{i\delta} + r'^4 e^{i2\delta} + r'^6 e^{i3\delta} + \dots \right\}$$

$$\boxed{E_{trans} = t' t E_0 \, e^{i k_f \frac{s}{2}} e^{i(k_1 x - \omega t)} \frac{1}{1 - r'^2 e^{i\delta}}}$$



$$= E_0 \underbrace{e^{i k_f \frac{s}{2}}} e^{i(k_1 x - \omega t)} \times \frac{1 - r^2}{1 - r^2 e^{i\delta}}$$

addition phase that comes from the fact that we are measuring the phase from point N

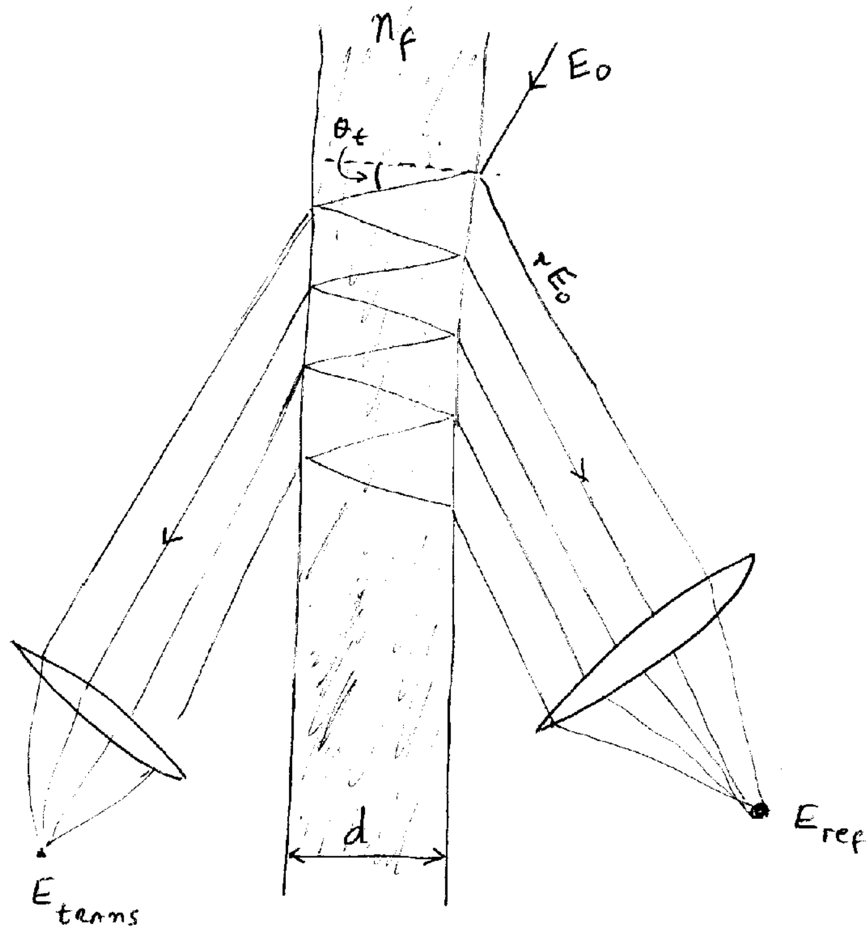
$$E_{trans} \cdot E^*_{transm} = E_0^2 \frac{(1-r^2)^2}{(1-r^2e^{i\delta})(1-r^2e^{-i\delta})}$$

$$= E_0^2 \frac{(1-r^2)^2}{1+r^4-r^2e^{i\delta}-r^2e^{-i\delta}}$$

$$= E_0^2 \frac{(1-r^2)^2}{1+r^4-2r^2\cos\delta} = E_0^2 \frac{(1-r^2)^2}{(1-r^2)^2+2r^2(1-\cos\delta)}$$

$$\boxed{|E_{transm}|^2 = E_0^2 \frac{(1-r^2)^2}{(1-r^2)^2+4r^2\sin^2\left(\frac{\delta}{2}\right)}}$$

$$= E_0^2 \frac{1}{1+F\sin^2\left(\frac{\delta}{2}\right)} \qquad F \equiv \frac{2r}{1-r^2}$$



$$|E_{ref}|^2 = E_0^2 \frac{F \sin^2\left(\frac{\delta}{2}\right)}{1 + F \sin^2\left(\frac{\delta}{2}\right)}$$

$$|E_{transm}|^2 = E_0^2 \frac{1}{1 + F \sin^2\left(\frac{\delta}{2}\right)}$$

$$F \equiv \frac{2r}{1-r^2}$$

$$\delta = 2 k_f d \cos\theta_t$$

- $|E_{trans}|^2$ is MAX when $\sin^2\left(\frac{\delta}{2}\right) = 0$

$$\boxed{\begin{array}{ll} |E_{transm}|^2 = E_0^2 & \text{when } \frac{\delta}{2} = m\pi \quad m = 0, 1, 2, \ldots \\ & \text{or} \\ |E_{refl}| = 0 & \delta = 0, 2\pi, 4\pi, 6\pi, \ldots \end{array}}$$

- $|E_{transm}|^2$ is min when $\sin^2\left(\frac{\delta}{2}\right) = 1$

$$\boxed{\begin{array}{ll} |E_{transm}|^2 = E_0^2 \dfrac{1}{1+F} & \text{when } \frac{\delta}{2} = \frac{2m+1}{2}\pi \\ & \text{or} \\ |E_{ref}|^2 = E_0^2 \dfrac{F}{1+F} & \delta = \pi, 3\pi, 5\pi, 7\pi \end{array}}$$

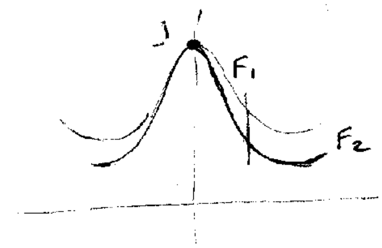
- $F = \left(\frac{2r}{1-r^2}\right)^2 \qquad \frac{dF}{dr} = 4F \frac{(1+r^2)}{(1-r^2)^2} > 0$

so, F increases as r increases

- $\frac{|E_{trans}|^2}{E_0^2} = \frac{1}{1+\alpha F} = I(F)$

$\frac{dI}{dF} = -\frac{\alpha}{(1+\alpha F)^2} < 0 \quad \forall F$ except when $\alpha = 0$

This means



$F_2 > F_1$

bell becomes sharper as F increases.