MAXWELL's EQUATIONS

ME in INTEGRAL FORM

Line integral

Surface integral

ME in DIFFERENTIAL FORM

operators GRADIENT
DIVERGENCE
ROTATIONAL

Gauss' theorem

Stoke's theorem

GENERATION, PROPAGATION and
DETECTION of ELECTROMAGNETIC WAVES

Maxwell Equations in differential form.—

First, some definitions in vector algebra:

- The operator ∇ (called "gradient")

$$\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

- ∇ acts on scalar fields

Example

If ϕ is the electric potential,

$$\nabla \phi = \left(\frac{\partial}{\partial x}\phi, \frac{\partial}{\partial y}\phi, \frac{\partial}{\partial z}\phi \right)$$

which, as we know, gives

$$= (-E_x, -E_y, -E_z)$$

Thus,

$$\nabla \phi = -\vec{E}$$

We see, when ∇ acts on a scalar field it gives a vector

- The divergence operator "$\nabla\cdot$"

Given a vector field $\vec{E}$,

$$\nabla\cdot\vec{E} \equiv \frac{\partial}{\partial x}E_x + \frac{\partial}{\partial y}E_y + \frac{\partial}{\partial z}E_z$$

$\nabla\cdot\vec{E}$ is a scalar quantity

- The rotational operator "$\nabla\times$"

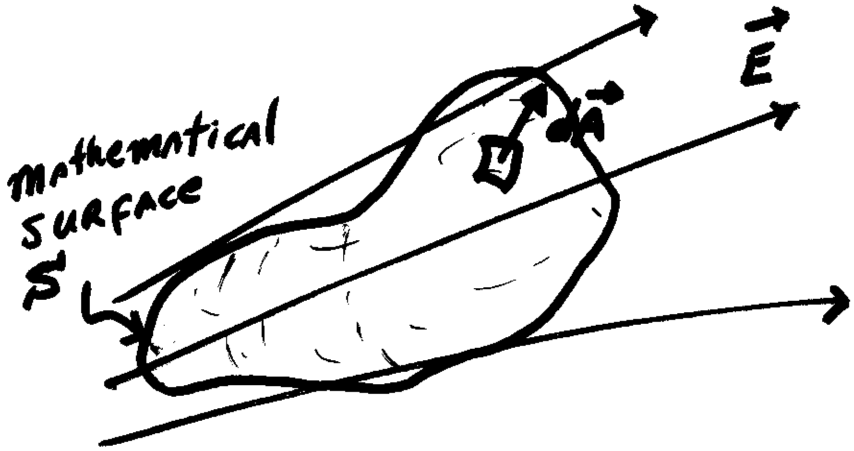
Given a vector field $\vec{E}$,

$$\nabla\times\vec{E} \equiv \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}E_z - \frac{\partial}{\partial z}E_y , \frac{\partial}{\partial z}E_x - \frac{\partial}{\partial x}E_z , \frac{\partial}{\partial x}E_y - \frac{\partial}{\partial y}E_x \right)$$

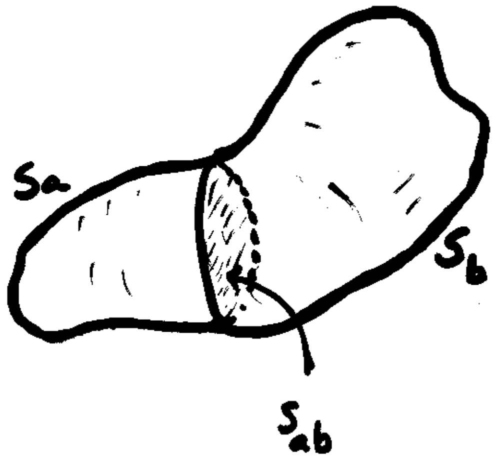
$\nabla\times\vec{E}$ is a vectorial quantity

• About the flux of a vector field



$$\Phi = \int_S \vec{E} \cdot d\vec{A}$$

Flux through the surface S



Now, surface S has been divided into two contiguous surfaces S_1 and S_2

$S_1 = S_a + S_{ab}$

$S_2 = S_b + S_{ab}$

Notice:

Flux through S =

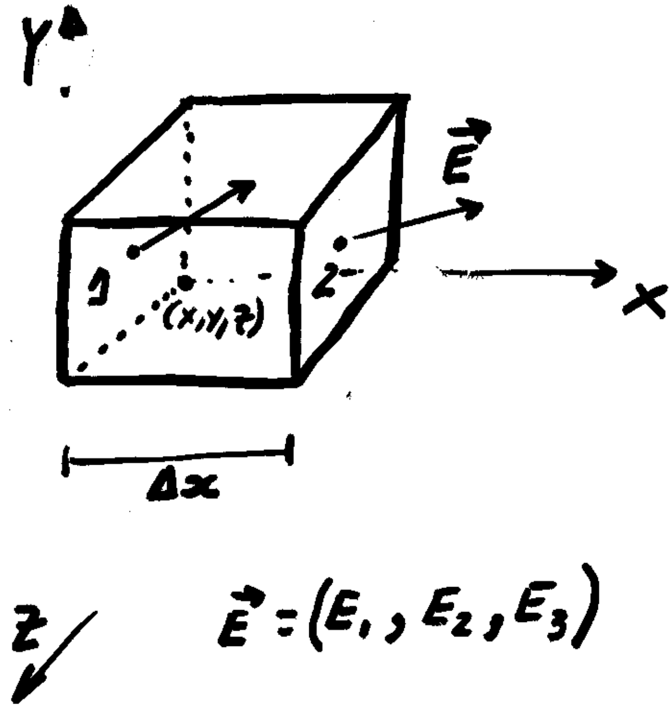
= flux through S_1

+

flux through S_2

$$\Phi_S = \Phi_{S_1} + \Phi_{S_2}$$

GAUSS' theorem



What is the flux through the cube of volume $\Delta x \Delta y \Delta z$?

$$\phi_{cube} = \phi_1 + \phi_2 + \ldots + \phi_6$$

$$\vec{E} = (E_1, E_2, E_3)$$

$$\phi_1 = -\underbrace{E_1(x)}\,\Delta y\,\Delta z$$

$$\phi_2 = \underbrace{E_1(x+\Delta x)}\,\Delta y\,\Delta z$$

$$\phi_1 + \phi_2 = \left[E_1(x+\Delta x) - E_1(x)\right]\Delta y\,\Delta z$$

$$= \left[\frac{\partial E_1}{\partial x}\Delta x\right]\Delta y\,\Delta z$$

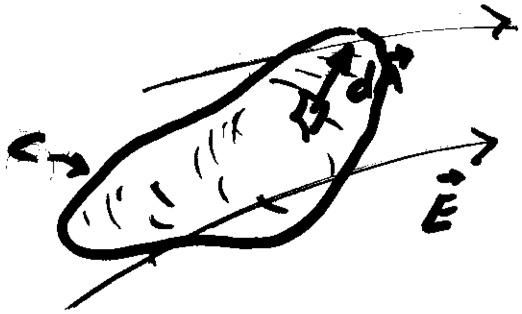
Similar expressions for $\phi_3 + \phi_4$ and $\phi_5 + \phi_6$

Thus,

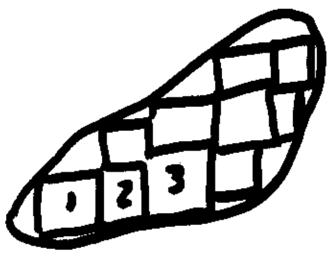
$$\phi_{cube} = \left[\frac{\partial E_1}{\partial x} + \frac{\partial E_2}{\partial y} + \frac{\partial E_3}{\partial z}\right]\Delta x\,\Delta y\,\Delta z$$

$$= \left[\nabla \cdot \vec{E}\right]\Delta V$$

flux through the cube of volume $\Delta V = \Delta x \Delta y \Delta z$



$$\Phi_S = \int_S \vec{E} \cdot d\vec{A}$$ surface integral



Notice, the volume inside the surface S can be diveded into many very small cubes

(we just need to make them very tiny; infinitesimal)

Each cube$_i$ has a surface A_i and a volume ΔV_i

We know that

$$\Phi_S = \Phi_{S_1} + \Phi_{S_2} + \cdots$$

$$= (\nabla \cdot \vec{E}) \Delta V_1 + (\nabla \cdot \vec{E}) \Delta V_2 + \cdots$$

which is nothing but

$$= \int_V \nabla \cdot \vec{E} \, dv$$ ← a volume integral

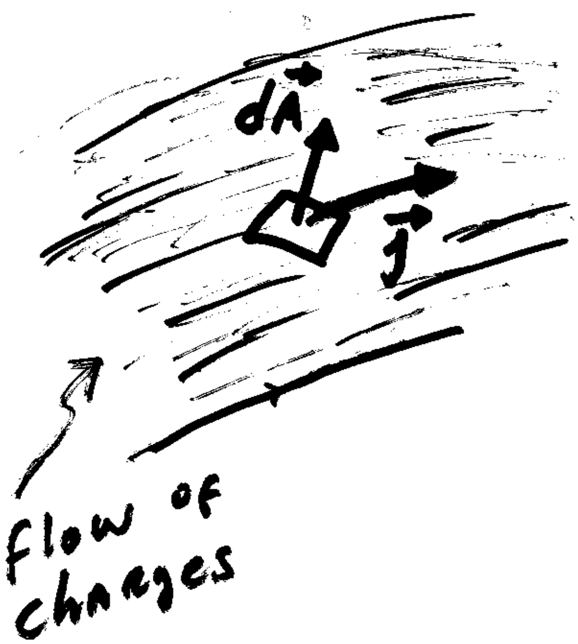
Thus, combining this expression with the one above, we have obtained

$$\int_S \vec{E} \cdot d\vec{A} = \int_V \nabla \cdot \vec{E} \, dv$$ GAUSS theorem

S is any mathematical closed surface

V is the volume inside S

Example: Expressing the charge conservation principle in differential form

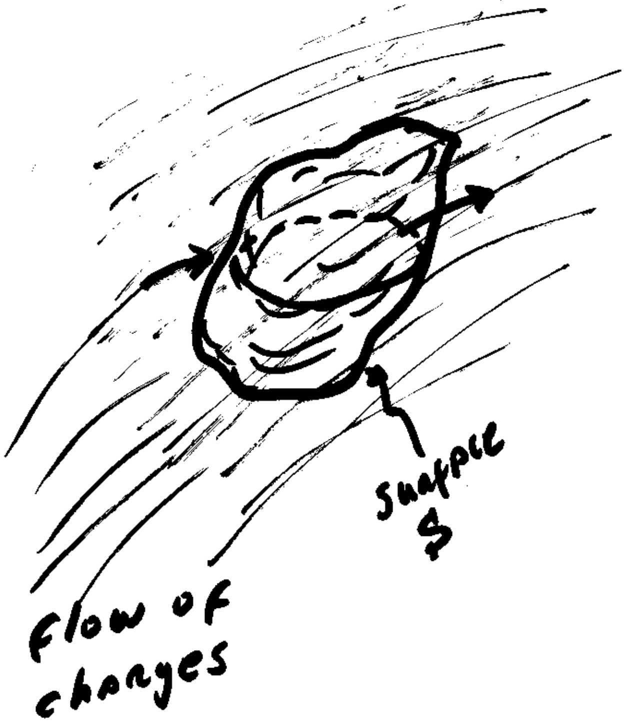


$$j = \frac{\text{current}}{\text{area}}$$

$\vec{j}$ points in the direction of the local velocity $\vec{v}$ of the charges

$$\text{current crossing } d\vec{A} = \vec{j} \cdot d\vec{A}$$

in units of coulomb/sec



$I =$ net current crossing the surface S

$$= \int_S \vec{j} \cdot d\vec{A}$$

I must be equal to the charge per unit time of the net charge lost inside the volume of the surface S

$$I = -\frac{d}{dt} Q_{inside}$$

$$= -\frac{d}{dt} \int_V \rho \, dv$$

thus,

$$\underbrace{\int_S \vec{j} \cdot d\vec{A}}_{\text{using Gauss' theorem}} = -\frac{d}{dt} \int_V \rho \, dv$$

where ρ is the charge density (coulomb/m^3)

$\rho = \rho(x, y, z, t)$

$$\int_S \nabla \cdot \vec{j} \, dV = -\frac{d}{dt} \int_V \rho \, dv \xrightarrow[\text{since the surface } S \text{ is stationary}]{=} -\int_V \frac{\partial \rho}{\partial t} \, dv$$

Since the surface S is arbitrary (it could even be infinitesimal), it must be that

$$\boxed{\nabla \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0}$$ charge conservation in differential form

<u>Example</u> First Maxwell Equation

$$\underbrace{\int_S \vec{E} \cdot d\vec{A}}_{\substack{\text{applying} \\ \text{Gauss theorem}}} = \frac{Q_{inside}}{\epsilon_0} = \frac{1}{\epsilon_0} \int_V \rho \, dv$$

$$\int_V \nabla \cdot \vec{E} \, dv = \int_V \frac{\rho}{\epsilon_0} \, dv$$

Since this is valid for any volume, including an infinitesimal one, it must be that

$$\boxed{\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}}$$ First's Maxwell Eq in differential form

<u>Example</u> Second Maxwell's Eq

$$\int_S \vec{B} \cdot d\vec{A} = 0$$

Applying Gauss theorem

$$\int_V \nabla\cdot\vec{B}\, dv = 0$$ for any arbitrary volume V

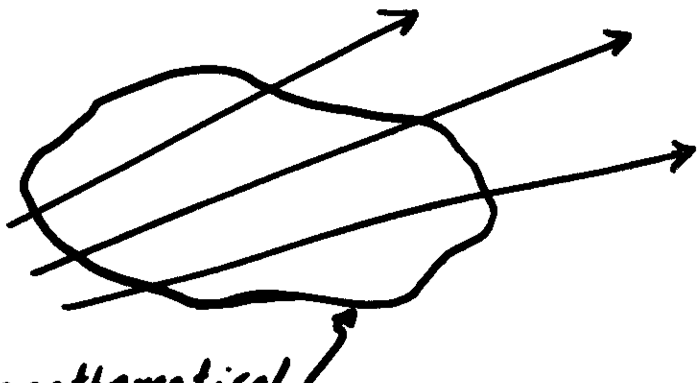
This implies

$$\boxed{\nabla\cdot\vec{B} = 0}$$

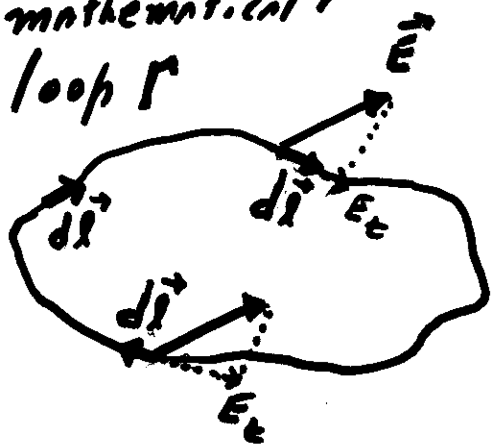
Second Maxwell's Eq in differential form

Stoke's theorem

About the circulation of a vector field $\vec{E}$

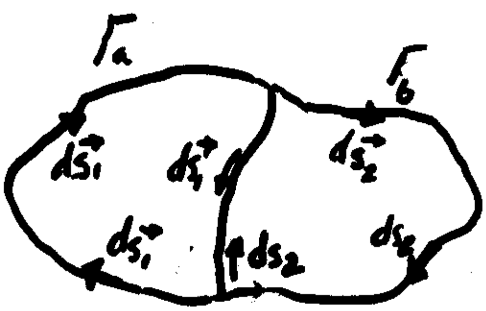


mathematical loop Γ



Circulation of $\vec{E}$ is the line integral of the tangential component of $\vec{E}$ around the loop Γ

$$\int E_t \, dl = \int \vec{E} \cdot d\vec{l}$$



Now, the loop Γ has been divided into two contiguous loops Γ_1 and Γ_2

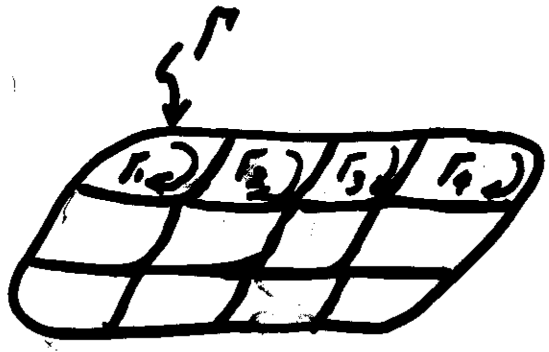
Notice:

$$\int_{\Gamma} = \int_{\Gamma_1} + \int_{\Gamma_2}$$

$\Gamma_1 = \Gamma_a + \Gamma_{ab}$

$\Gamma_2 = \Gamma_b + \Gamma_{ab}$

Generalization



$$\int_{\Gamma} = \int_{\Gamma_1} + \int_{\Gamma_2} + \dots$$

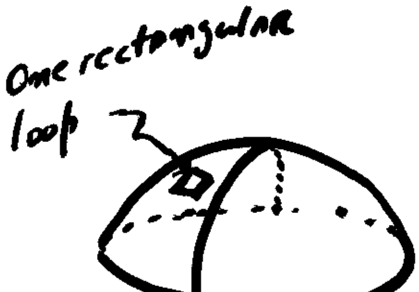
Here we have assumed that the different loops $\Gamma_1, \Gamma_2, \dots$ are in the plane of Γ.

But it doesn't have to be that way

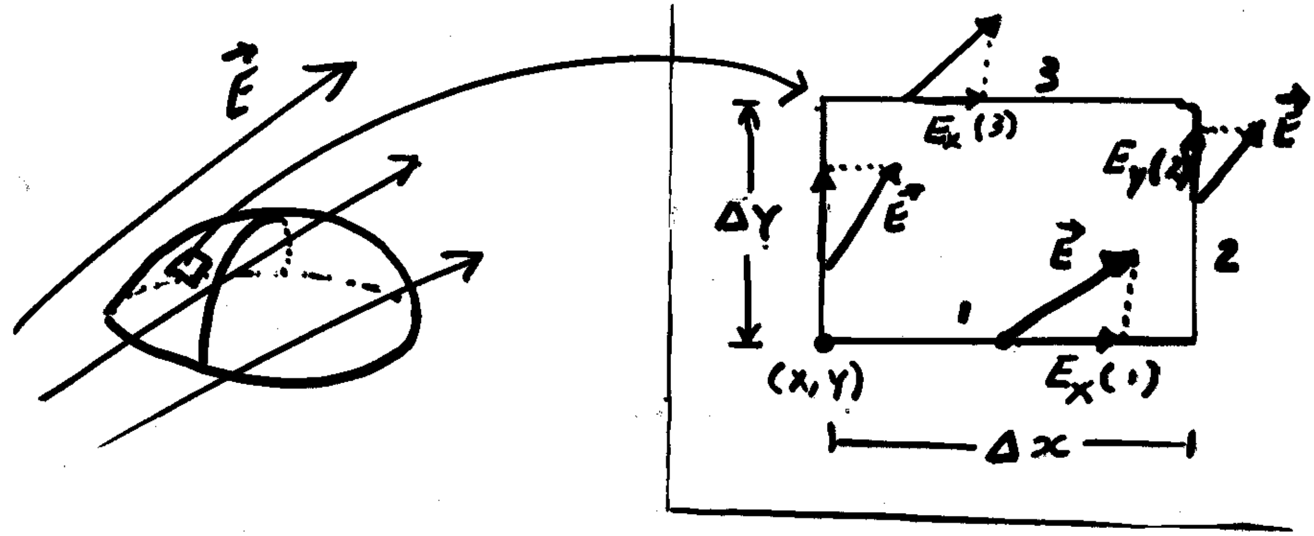
The small loops can lie on any surface having Γ as its boundary

By choosing the loops small enough, each can be considered a flat rectangular loop.

Circulation around a rectangular loop



For the particular small rectangular loop of area $\Delta x \Delta y$ shown in the figure, let's choose a reference such that the loop results lying at the XY plane



If the result of the circulation can later be put in vectorial notation, then it will be the same no matter how the axis XYZ were chosen

$$\int_{\square} \vec{E} \cdot d\vec{s} = E_x(1)\,\Delta x + E_Y(2)\,\Delta Y +$$
$$- E_x(3)\,\Delta x - E_Y(4)\,\Delta Y$$

Since $E_x(3) - E_x(1) \approx \dfrac{\partial E_x}{\partial Y}\,\Delta Y$

$$E_Y(2) - E_Y(4) \approx \frac{\partial E_Y}{\partial x}\,\Delta x$$

$$\int_{\square} \vec{E} \cdot d\vec{s} = \underbrace{\left(\frac{\partial E_Y}{\partial x} - \frac{\partial E_x}{\partial Y}\right)}\,\Delta x\,\Delta Y$$

z-component of $\nabla \times \vec{E}$

$$\oint_{\square} \vec{E}\cdot d\vec{s} = \left(\nabla \times \vec{E}\right)_z \Delta x \, \Delta y$$

We can express this result in vectorial form by realizing that z component is the direction perpendicular to the small loop

$$= \left(\nabla \times \vec{E}\right)_n \Delta x \Delta y = \left(\nabla \times \vec{E}\right)\cdot \hat{n} \, \Delta x \, \Delta y$$

$$\oint_{\square} \vec{E}\cdot d\vec{s} = \left(\nabla \times \vec{E}\right)\cdot d\vec{A}$$

$d\vec{A} = \hat{n}\,\Delta x \Delta y$

We extend this result to multi-connected loops

$$\boxed{\oint_{\Gamma} \vec{E}\cdot d\vec{l} = \int_{S} \left(\nabla \times \vec{E}\right)\cdot d\vec{A}}$$

Stoke's theorem

S is any surface whose boundary is Γ

Example: Third Maxwell Eq

$$\underbrace{\int_\Gamma \vec{E}\cdot d\vec{l}}_{\text{Applying Stoke's theorem}} = -\underbrace{\frac{d}{dt}\int_S \vec{B}\cdot d\vec{A}}_{\text{if the surface } S \text{ is stationary}}$$

$$\int_S (\nabla \times E)\cdot d\vec{A} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}$$

this expression is valid for ~~any~~ arbitrary surface, including a infinitesimal rectangle. Thus

$$\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

Third Maxwell Eq in differential form

Example: Fourth Maxwell's Eq

$$\underbrace{\int_\Gamma \vec{B}\cdot d\vec{l}}_{\text{Applying Stoke's theorem}} = \mu_0 \int_S \vec{j}\cdot d\vec{A} + \epsilon_0\mu_0 \underbrace{\frac{d}{dt}\int_S \vec{E}\cdot d\vec{A}}_{\text{If } S \text{ is stationary}}$$

$$\int_S (\nabla \times \vec{B})\cdot d\vec{A} = \mu_0 \int_S \vec{j}\cdot d\vec{A} + \epsilon_0\mu_0 \int_S \frac{\partial \vec{E}}{\partial t}\cdot d\vec{A}$$

Since this expression is valid for any arbitrary surface S, then

$$\boxed{\nabla \times \vec{B} = \mu_0 \vec{j} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}}$$

4rth Maxwell's Eq
in differential form