MAXWELL's EQUATIONS

ME in INTEGRAL FORM

Line integral

Surface integral

ME in DIFFERENTIAL FORM

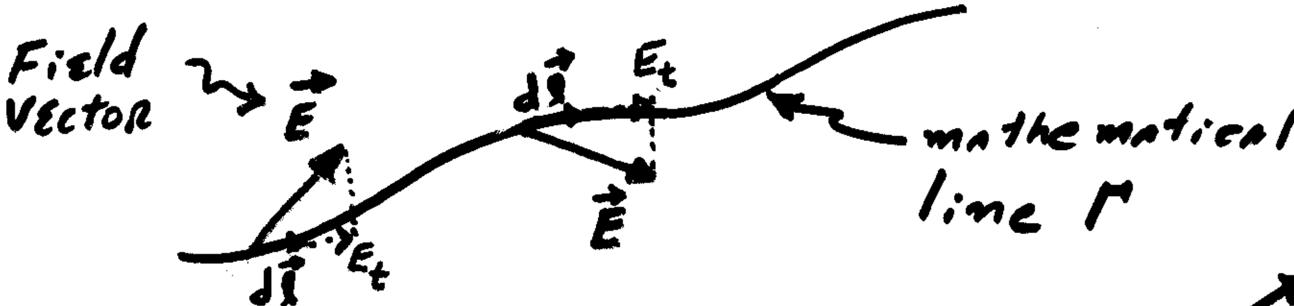
operators GRADIENT
DIVERGENCE
ROTATIONAL

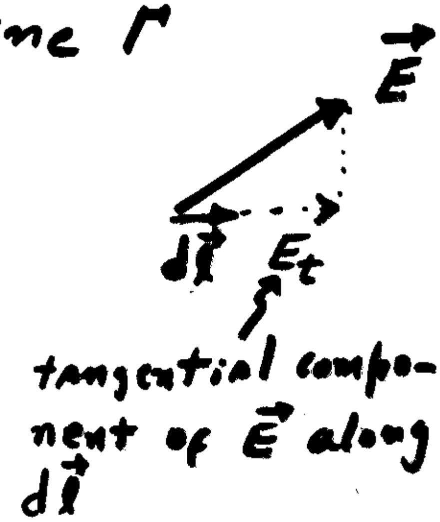
Gauss' theorem

Stoke's theorem

GENERATION, PROPAGATION and
DETECTION of ELECTROMAGNETIC WAVES

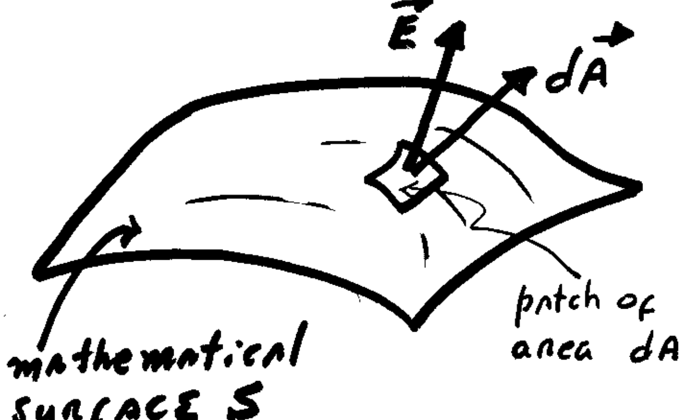
Line integral





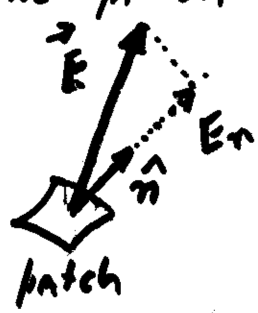
$$\int_\Gamma E_t \, dl = \int_\Gamma \vec{E} \cdot d\vec{l}$$

Surface integral



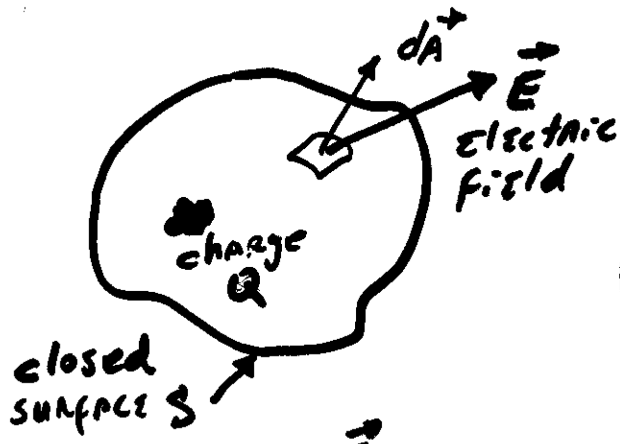
$$d\vec{A} = \hat{n} \, dA$$

where
$\hat{n}$ is the unit vector perpendicular to the patch



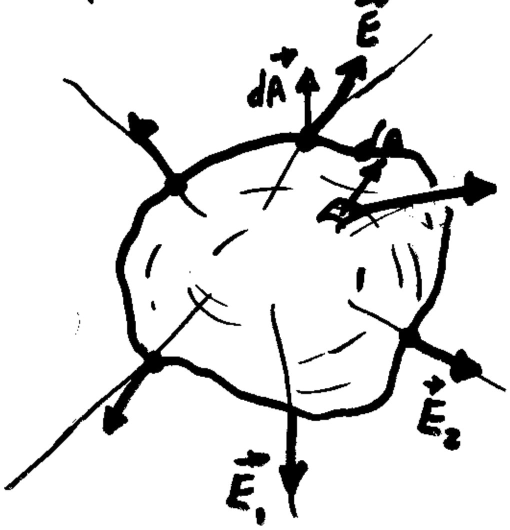
$$\int_S E_n \, dA = \int_S \vec{E} \cdot \hat{n} \, dA = \int_S \vec{E} \cdot d\vec{A}$$

First Maxwell Eq



$$\boxed{\int_S \vec{E}\cdot d\vec{A} = \frac{Q_{inside}}{\epsilon_0}}$$

ϵ_0: permitivity of free-space



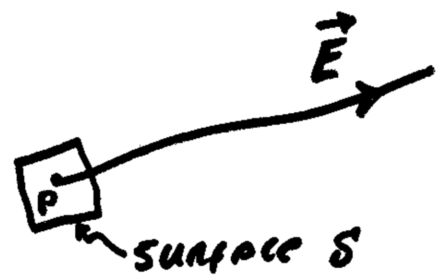
$\vec{E}$ is evaluated at each point ON the surface S

electric flux $\Phi_E = \int_S \vec{E}\cdot d\vec{A}$

Notice:

If an electric field line originates at a given point P,
then
there must exist charge at that point

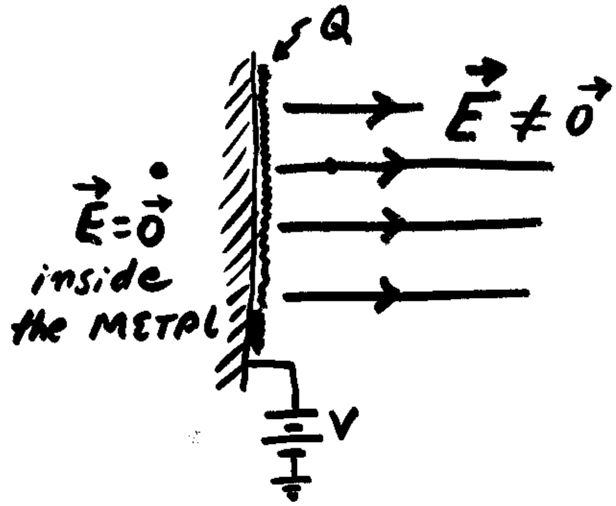
Proof



obviously $\Phi_E \neq 0$

Therefore, according to the 1st ME, there must be charge at P

This is what happens in metals, for example.

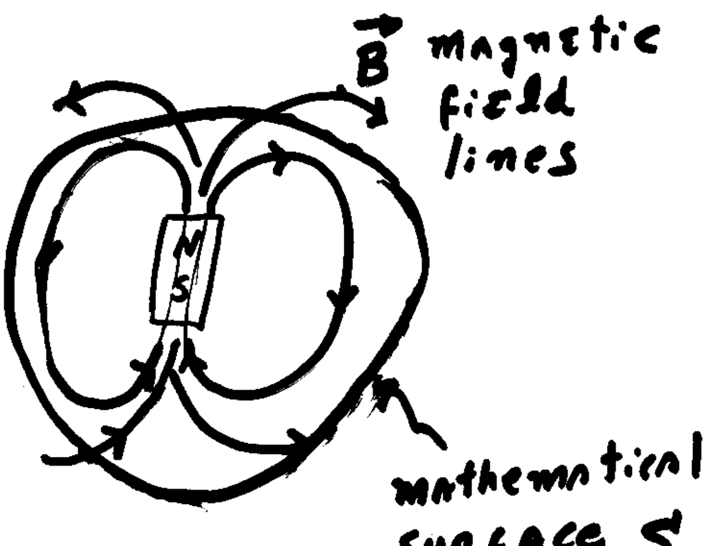


When charges are present:

lines of $\vec{E}$ originate on positive charges and terminate on negative charges

Everywhere else the $\vec{E}$ lines can twist and turn in space, but they cannot start or stop

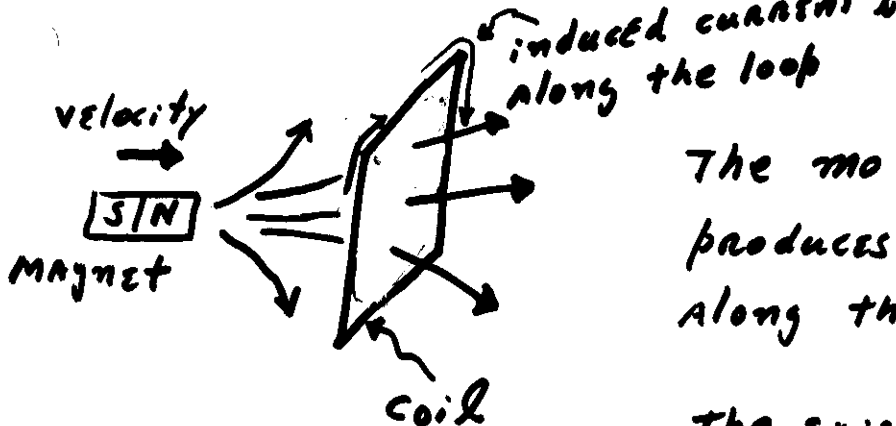
SECOND MAXWELL Eq.



$$\boxed{\int_S \vec{B} \cdot d\vec{A} = 0}$$

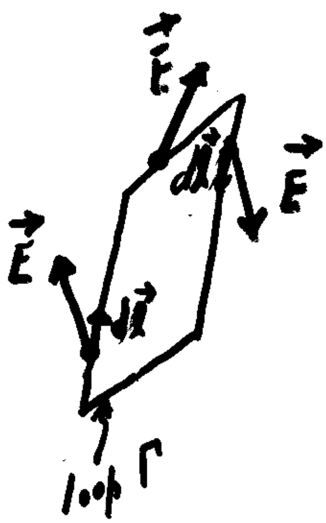
No magnetic monopoles have been observed (so far)

Third Maxwell Eq



The motion of the magnet produces an induced current along the coil

The existance of a current implies the presence of a electric field or (equivalently) an electromotive force



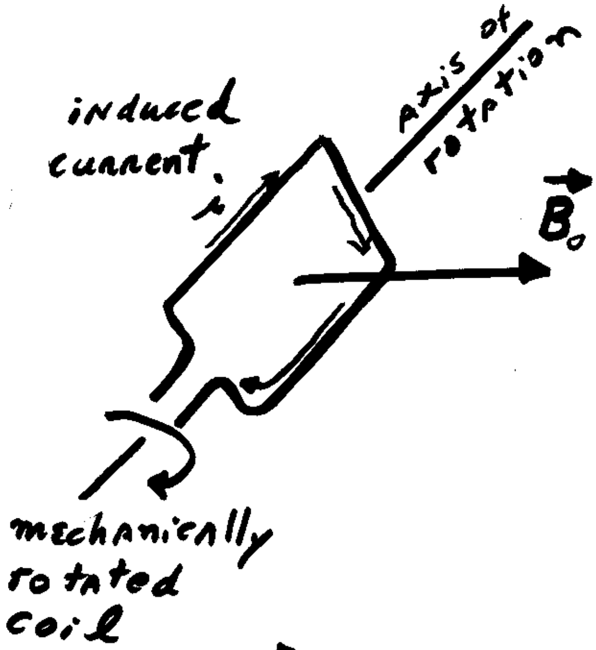
$$\underset{\text{electromotive force}}{\mathcal{E}} = \int_{\substack{\text{loop} \\ \Gamma}} \vec{E} \cdot d\vec{l} \qquad (= Ri)$$

But, what is the value of $\mathcal{E}$?
(1v, 2v, −2.3v … ?)

$$\boxed{\mathcal{E} = \int_{\substack{\text{loop} \\ \Gamma}} \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\left(\int_S \vec{B} \cdot d\vec{A}\right)}$$

where S is any <u>open</u> surface having the loop Γ as its boundary

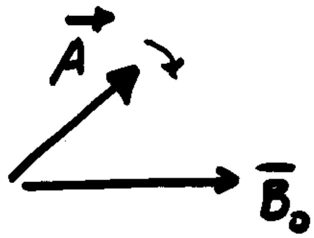
$$\mathcal{E} = -\frac{d}{dt}\Phi_M$$



Notice:

A $\vec{B} = \vec{B}(t)$ ensures the induction of an electric field $\vec{E}$ and, consequently, a current i along a coil.

However, a static magnetic field $\vec{B}_o$ can also be exploited to generate currents along a coil (see figure).



$\vec{A}$ revolves around $\vec{B}_o$ causing a time dependent magnetic flux Φ_M

So, to have an electromotive force what matters is to have time dependent $\Phi_M(t)$

[In the figure above, since there is not an induced electric field, how can we have an electromotive force $\mathcal{E} = \int \vec{E} \cdot d\vec{s}$? Shouldn't $\vec{E}$ be zero because $\vec{E} = \vec{0}$?

Answer: We better re-define $\mathcal{E}$ as

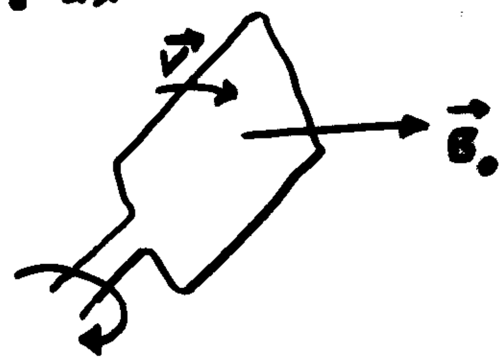
$$\mathcal{E} = \int_{\text{loop } \Gamma} \frac{\text{Magnetic Electric force on } q}{q} \cdot d\vec{l}$$

For the particular case of a mechanically rotated coil immersed in a uniform magnetic field $\vec{B}_0$ (see previous figure)

$$\mathcal{E} = \int_{\text{loop } \Gamma} \frac{\text{magnetic force on } q}{q} \cdot d\vec{l}$$

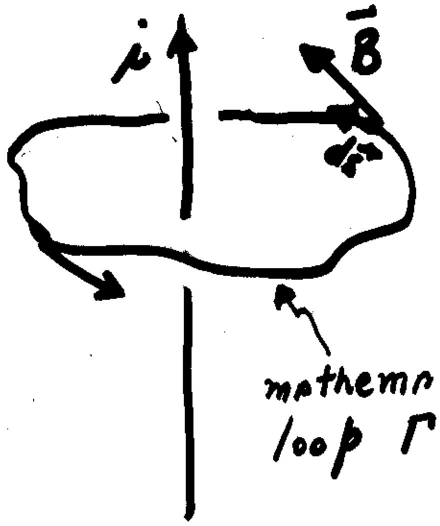
$$= \int_{\text{loop } \Gamma} \frac{q \vec{v} \times \vec{B}_0}{q} \cdot d\vec{l}$$

$$= \int_{\text{loop } \Gamma} (\vec{v} \times \vec{B}_0) \cdot d\vec{l}$$



4th Maxwell's Eq.

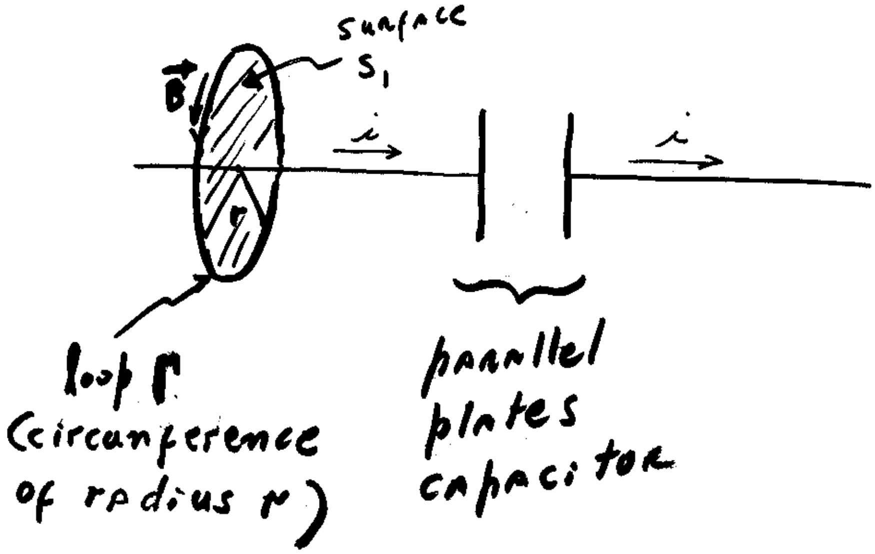
It built upon Ampere's Law



$$\int_{\text{loop } \Gamma} \vec{B} \cdot d\vec{s} = \mu_0 \underbrace{i}_{\text{current enclosed by the loop } \Gamma}$$

Maxwell notice this law was incomplete

James Clerk Maxwell noticed there was something wrong with the fourth equation. For example:



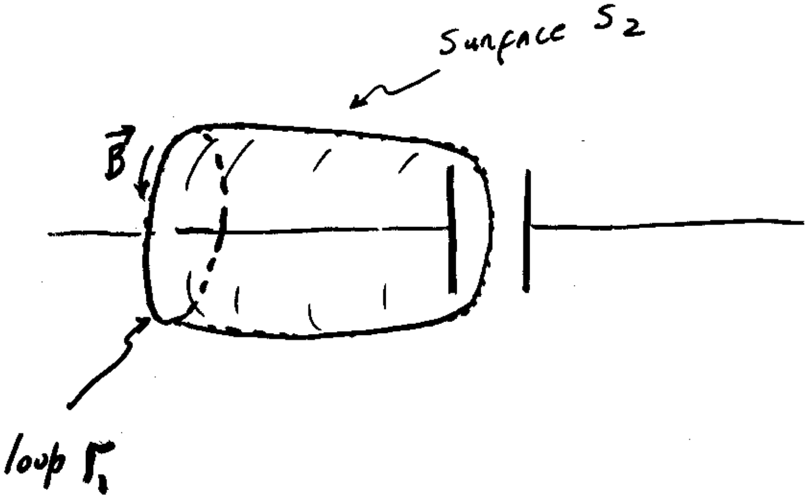
$$\int_{\Gamma} \vec{B} \cdot d\vec{l} = \mu_0 \underbrace{i}_{\text{current crossing the surface } S_1}$$

$$B \, 2\pi r = \mu_0 i$$

$$\Rightarrow$$

So, $B = \dfrac{\mu_0}{2\pi r} i$ ①

On the other hand,



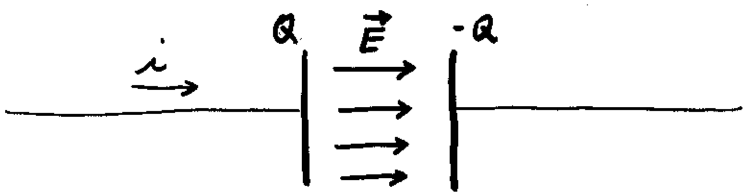
$$\int_{\Gamma} \vec{B} \cdot d\vec{l} = \mu_0 \underbrace{0}_{\text{current crossing the surface } S_2}$$

$B \, 2\pi r = \mu_0 \times 0$ ②

So, $B = 0$

How can be possible that ① and ② give different results for the same B?
Something must be wrong with the Ampere's law

In order to solve this contradictory situation, let's take a look to what is going on inside the parallel plates capacitor

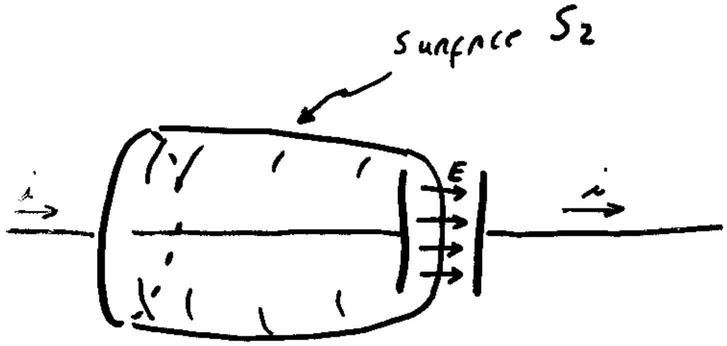


Let's remember

$$E = \frac{Q}{A\,\epsilon_0}$$

(A is the area of the plates)

The current i continuously accumulates charge on the plates, this is $Q = Q(t)$



Notice, the electric flux Φ_E crossing the surface S_2 is given by

$$\Phi_E = EA = \frac{Q}{\epsilon_0}$$

or

$$Q = \epsilon_0 \Phi_E$$

From the last expression, we obtain

$$\underbrace{\frac{dQ}{dt}}_{\text{but, this is equal to } i} = \epsilon_0 \frac{d\Phi_E}{dt}$$

$$\boxed{i = \epsilon_0 \frac{d\Phi_E}{dt}}$$

So, an electric flux that changes with time is "equivalent" to a current

Maxwell proposed the following modified version of the Ampere's law

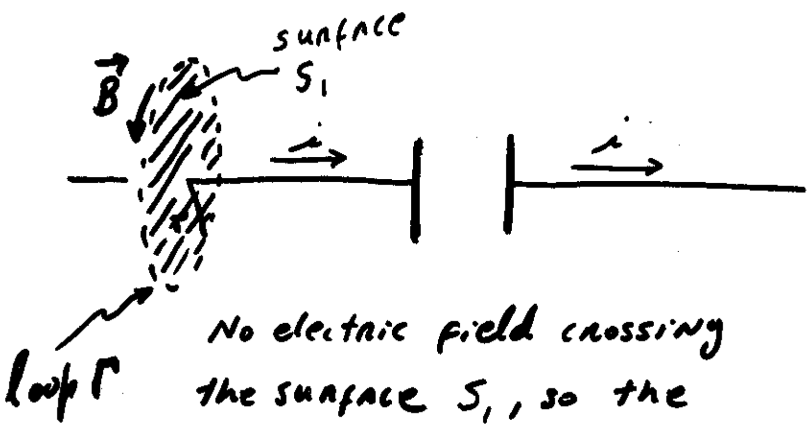
$$\int_{\text{loop } \Gamma} \vec{B}\cdot d\vec{s} = \mu_0 i + \underbrace{\mu_0 \epsilon_0 \frac{d\Phi_E}{dt}}_{\text{called "displacement current" } i_d}$$

or, written more explicitly

$$\boxed{\int_{\text{loop } \Gamma} \vec{B}\cdot d\vec{l} = \mu_0 \underbrace{i}_{\text{current enclosed by } \Gamma} + \mu_0 \epsilon_0 \frac{d}{dt}\int_{\text{surface } S} \vec{E}\cdot d\vec{A}}$$

Fourth Maxwell equation (1873)

Going back to our example, let's apply the 4th Maxwell equation and find $\vec{B}$ using the surface S_1



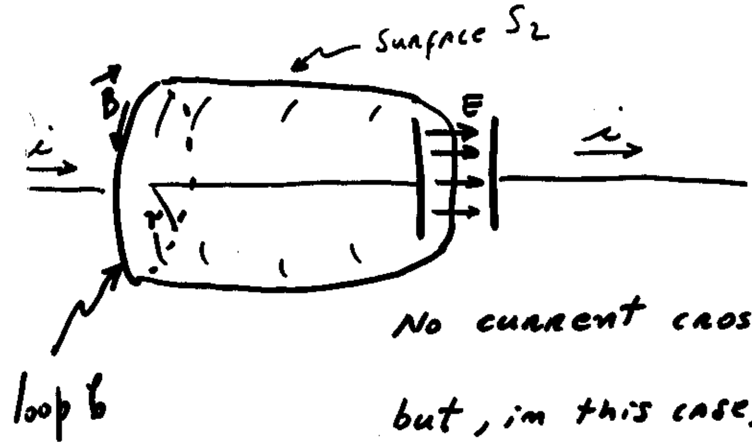
No electric field crossing the surface S_1, so the 4th Maxwell equation takes the form:

$$\int_{\text{loop } \Gamma} \vec{B}\cdot d\vec{l} = \mu_0 i$$

From which we obtain

$$B = \frac{\mu_0 i}{2\pi r}$$

What if we choose the surface S_2 ?



No current crossing the surface S_2 ,

but, in this case, there does exist a electric field $\vec{E}$ crossing the surface.

So, the 4th Maxwell equation takes the form:

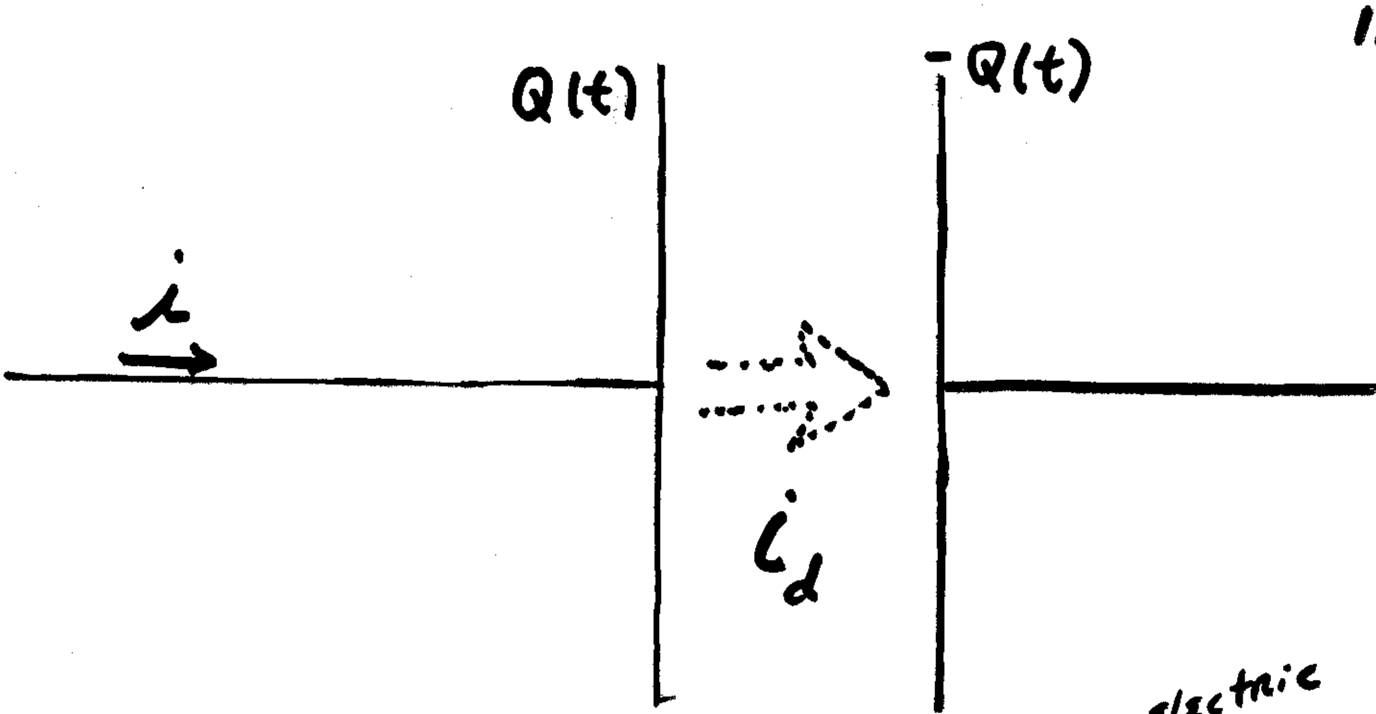
$$\int_{\text{loop } \Gamma} \vec{B} \cdot d\vec{l} = \mu_0 \epsilon_0 \frac{d}{dt} \underbrace{\int_{\substack{\text{surface} \\ S_2}} \vec{E} \cdot d\vec{A}}_{= EA = \frac{Q}{A \epsilon_0} A = \frac{Q}{\epsilon_0}}$$

$$= \mu_0 \epsilon_0 \frac{d}{dt}\left(\frac{Q}{\epsilon_0}\right) = \mu_0 \underbrace{\frac{dQ}{dt}} = \mu_0 i$$

$$\int_{\text{loop } \Gamma} \vec{B} \cdot d\vec{l} = \mu_0 i$$

From which we obtain

$$B = \frac{\mu_0}{2\pi r} i$$ ← Same result as ④ when the surface S_1 was used.



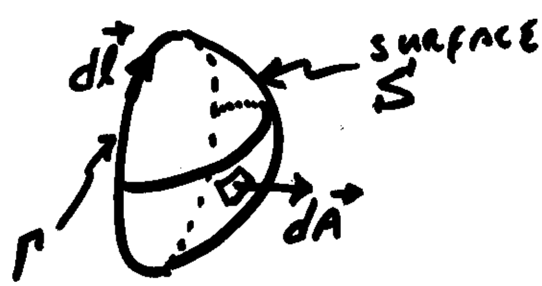
$$i = \frac{dQ}{dt}$$ charge on the plate

$$i_d = \epsilon_0 \frac{d}{dt}\Phi_E$$ electric flux

"displacement current"

More general,

$$\int_\Gamma \vec{B}\cdot d\vec{l} = \mu_0 \int_S \vec{j}\cdot d\vec{A} + \mu_0 \epsilon_0 \frac{d}{dt}\int_S \vec{E}\cdot d\vec{A}$$



4th MAXWELL's equation