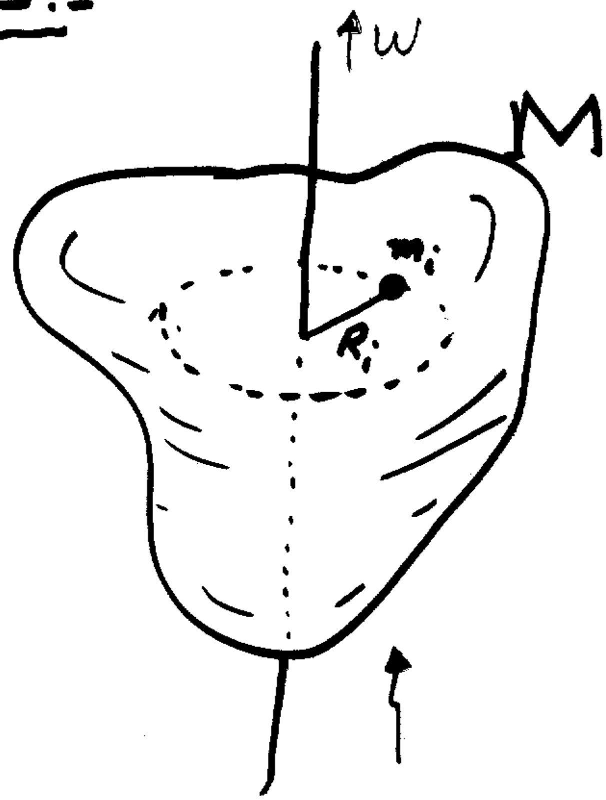
MOMENTUM OF INERTIA OF SOME SYMMETRIC RIGID BODIES.-

$$K_{ROT} = \sum_i \frac{1}{2} m_i v_i^2$$ ROTATIONAL KINETIC ENERGY

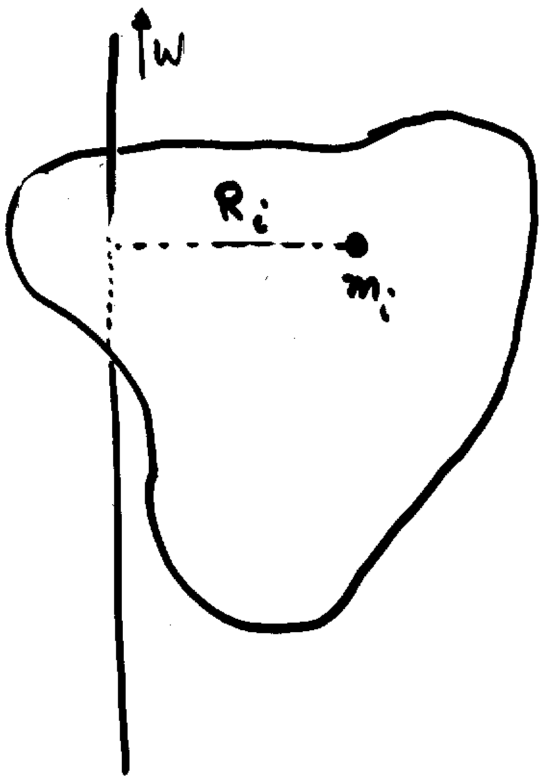
$$= \frac{1}{2} \underbrace{\left(\sum_i m_i R_i^2 \right)}_{I} \omega^2$$

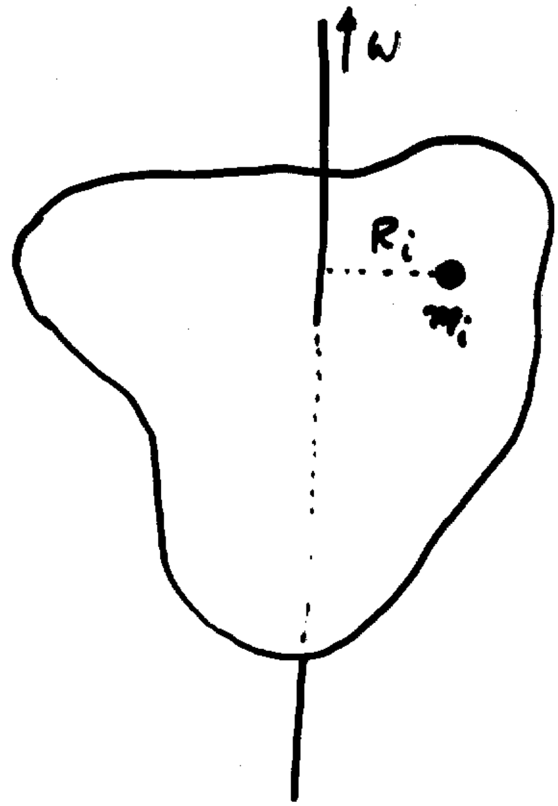
geometric factor associated with a given rigid body. It is called MOMENTUM OF INERTIA



ROTATION AROUND A FIXED AXIS

Notice: To evaluate the rotational kinetic energy what matters is the distance R_i of the individual masses m_i that make up the rigid body to the axis of rotation.





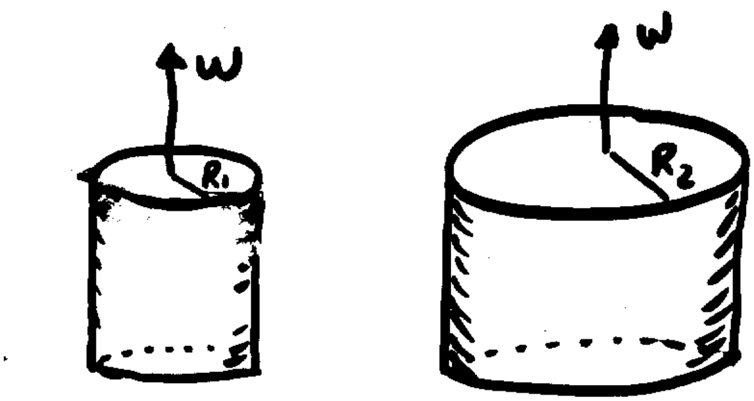
$$K_{rot} = \frac{1}{2}\underbrace{\left(\sum_i m_i R_i^2\right)}_{I}\omega^2 \qquad K_{rot} = \frac{1}{2}\underbrace{\left(\sum_i m_i R_i^2\right)}_{I'}\omega^2$$

$$I \neq I'$$

Notice: When talking about momentum of inertia, we have to specify with respect to what axis is the momentum of inertia being evaluated

QUESTION: the cylinders shown in the figure are rotating with the same angular velocity $\omega = 20$ radians/sec. a) Which cylinder has higher angular momentum?

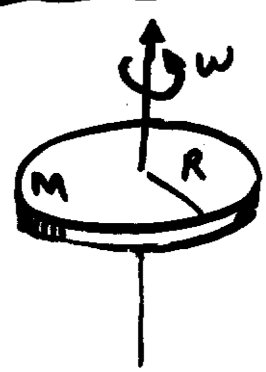
b) Which one has higher momentum of inertia?



$M = 3\,Kg$ $\qquad$ $M = 3\,Kg$

$R_1 = 0.1\,m$ $\qquad$ $R_2 = 0.5\,m$

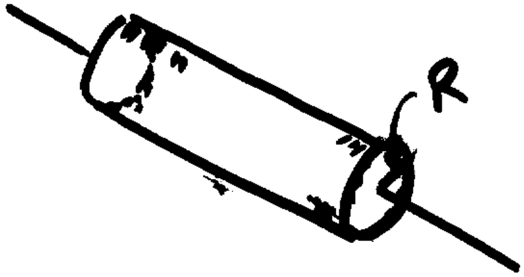
MOMENTUM OF INERTIA OF SOME RIGID BODIES:



$I = \frac{1}{2} M R^2$ $\qquad$ $I = \frac{1}{4} M R^2$

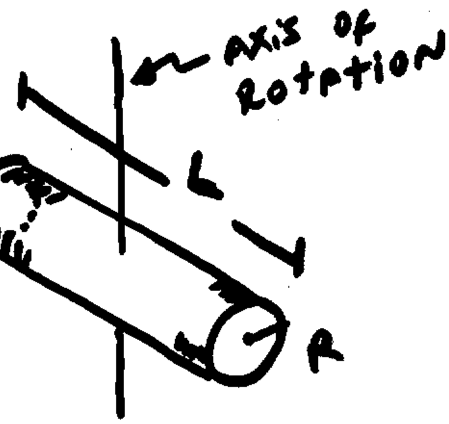
DISK $\qquad$ DISK

of radius R and mass M $\qquad$ of radius R and mass M



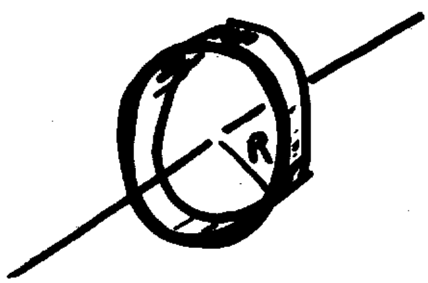
$$I = \frac{1}{2} M R^2$$

cylinder



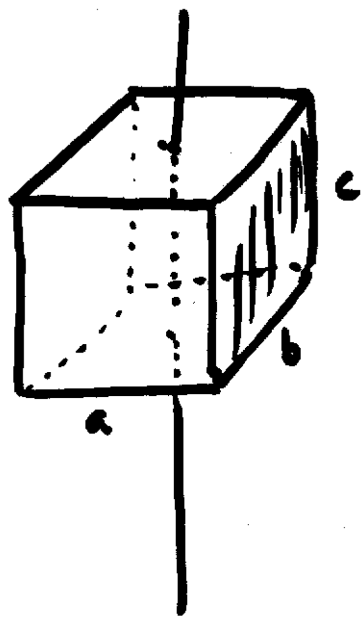
$$I = \frac{1}{2} M \left(\frac{R^2}{2} + \frac{L^2}{6} \right)$$

cylinder



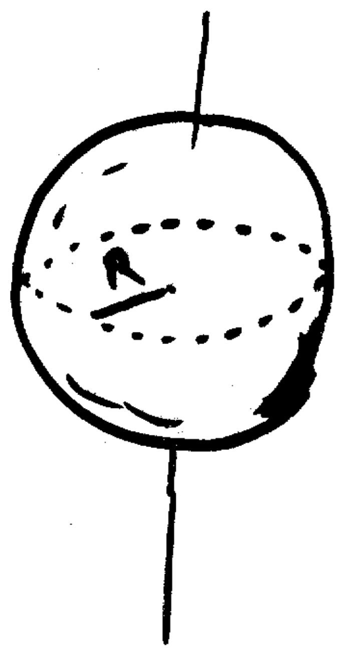
$$I = M R^2$$

Ring



$$I = \frac{1}{12} M (a^2 + b^2)$$

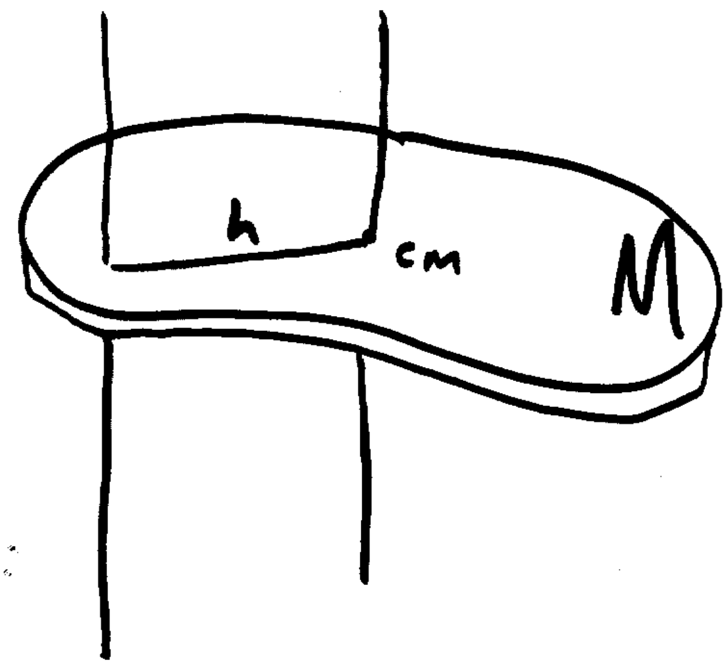
Parallelepiped



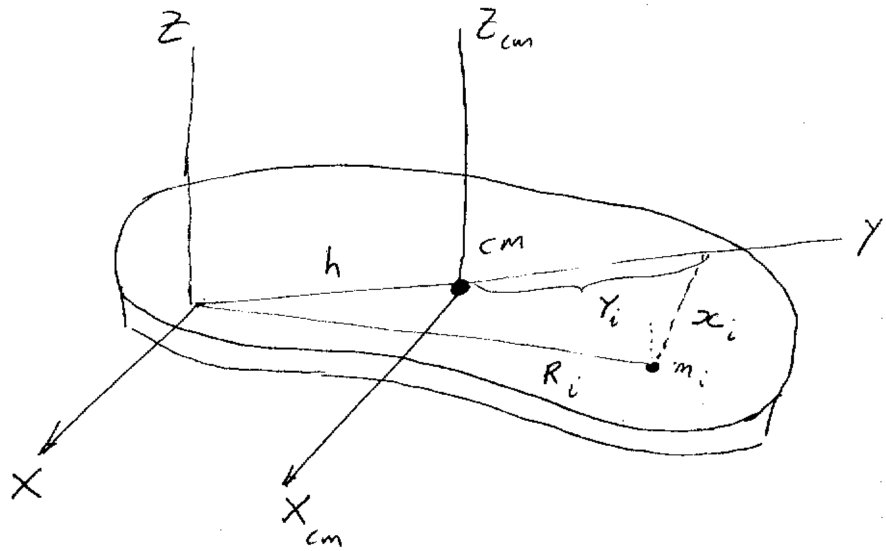
$$I = \frac{2}{5} M R^2$$

sphere

Parallel-axis theorem



$$I = I_{cm} + M h^2$$



$$I = \sum_i m_i R_i^2$$

$$= \sum_i m_i \left[x_i^2 + (y_i + a)^2\right]$$

$$= \sum_i m_i \left(x_i^2 + y_i^2 + a^2 + 2a y_i\right)$$

$$= \underbrace{\sum_i m_i (x_i^2 + y_i)}_{I_{cm}} + \sum_i m_i a^2 + \sum_i m_i 2a y_i$$

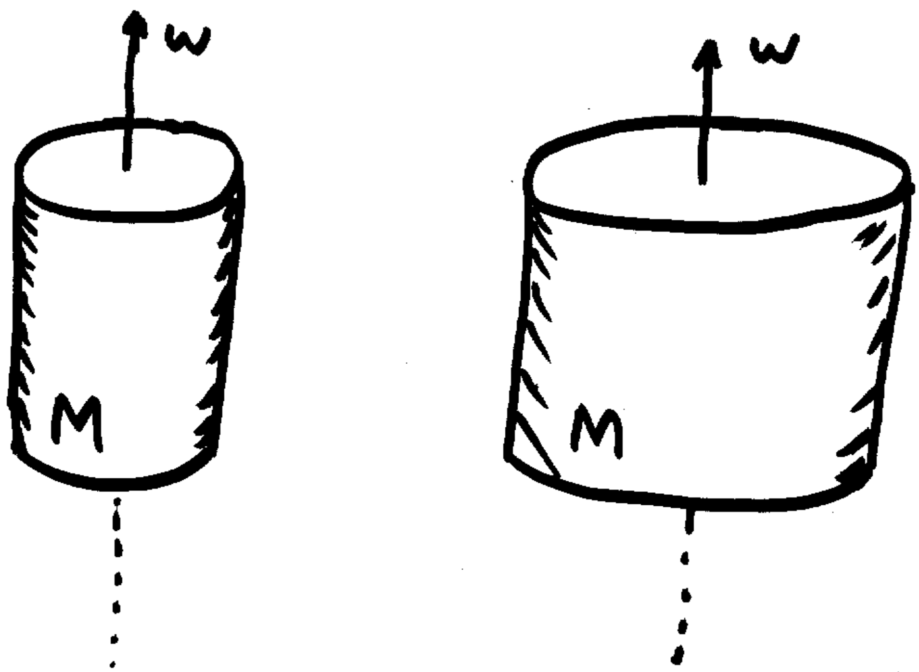
$$= I_{cm} + a^2 \sum_i m_i + 2a \underbrace{\sum_i m_i y_i}$$

Y coordinate of the cm with respect to the cm therefore it is equal to zero

Thus

$$\boxed{I = I_{cm} + M a^2}$$

WORK and KINETIC ENERGY



We know it costs energy to put a system into motion.

The systems above were at rest, but now they are rotating with the same angular velocity ω (around one symmetry axis)

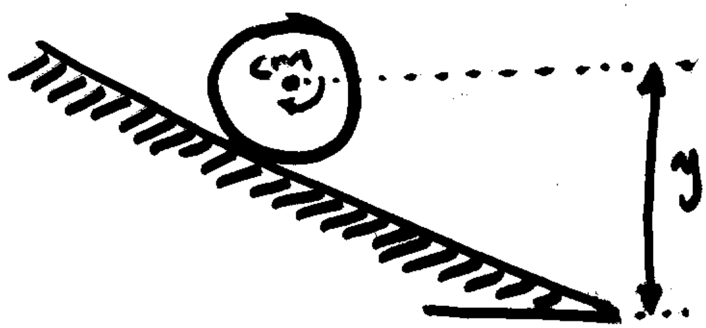
QUESTION: which cylinder has more kinetic energy?

which cylinder required more work to be put into motion?

$$\Delta K = W$$

In particular, if the system moves under the action of a conservative force, like the gravitational force for example, we will obtain

$$K_f + mgY_f = K_i + mgY_i$$



$$\boxed{E = K_f + mgY}$$

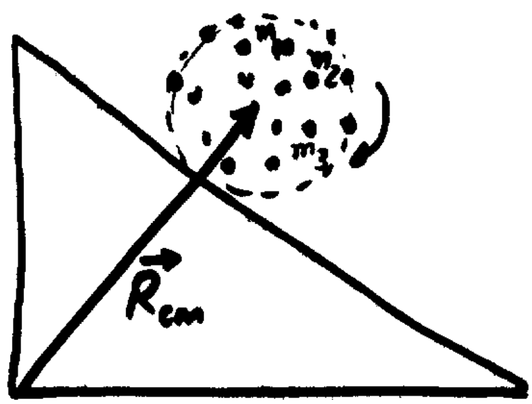
mechanical energy is conserved

$$\boxed{E = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I\omega^2 + mgy}$$

E is conserved throughout the motion

APPENDIX

CONSERVATION of the MECHANICAL ENERGY Rigid body motion



$$\vec{F}_{total} = \vec{N} + \vec{F}_f + \sum_i m_i \vec{g} + \underbrace{\sum_{ij} \vec{F}_{ij}}_{\text{internal forces}}$$

while the ball is rolling down, what is the work done by all these?

$\vec{N}$:
- It acts on a different m_i each time
- It acts on the so called "point of contact."
- The only possible displacement of the point of contact would be perpendicular to $\vec{N}$. Therefore the work done by $\vec{N}$ is ZERO

$\vec{F}_f$ - It acts on the rolling ball through the instantaneous point of contact (ipc)

- If we assume no slipping, the instantaneous velocity of the "ipc" is zero

Therefore there will not be kinetic friction. ⇔ no work

There is static friction but such force does not produce work.

The only forces that will produce work are: the gravitational force and the internal forces.

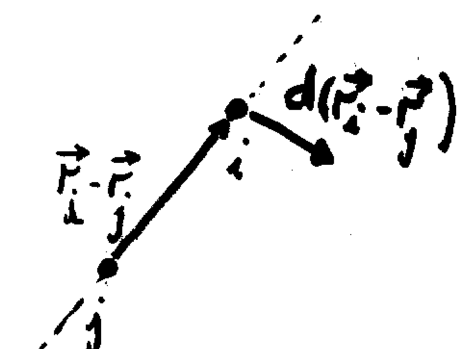
To evaluate this work, let's consider two arbitrary "small" masses m_i and m_j.

$$\underbrace{\vec{F}_{ij}\cdot d\vec{r}_i + \vec{F}_{ji}\cdot d\vec{r}_j} = \vec{F}_{ij}\cdot d\vec{r}_i + (-\vec{F}_{ij})\cdot d\vec{r}_j$$

$$= \vec{F}_{ij}\cdot d(\vec{r}_i - \vec{r}_j)$$

Since we are dealing with a rigid body, the magnitude of the vector $(\vec{r}_i - \vec{r}_j)$ remains const. throughout the motion

Then, the only possible orientation of $d(\vec{r}_i - \vec{r}_j)$ is to be perpendicular to the line that passes through m_i and m_j

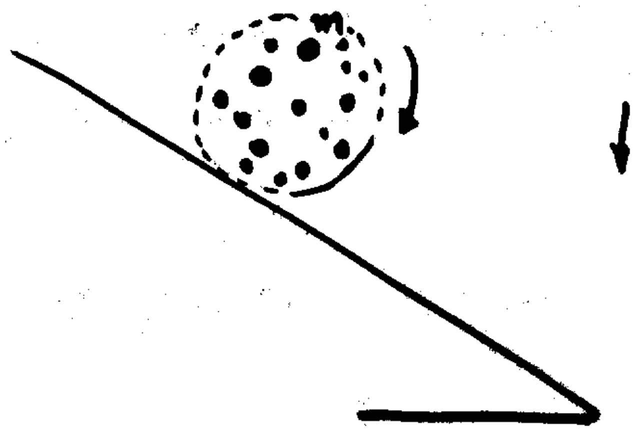


Therefore, $\vec{F}_{ij}\cdot d(\vec{r}_i - \vec{r}_j) = 0$

If we extend this argument to all pair of particles (contained in the rolling block) we will conclude:

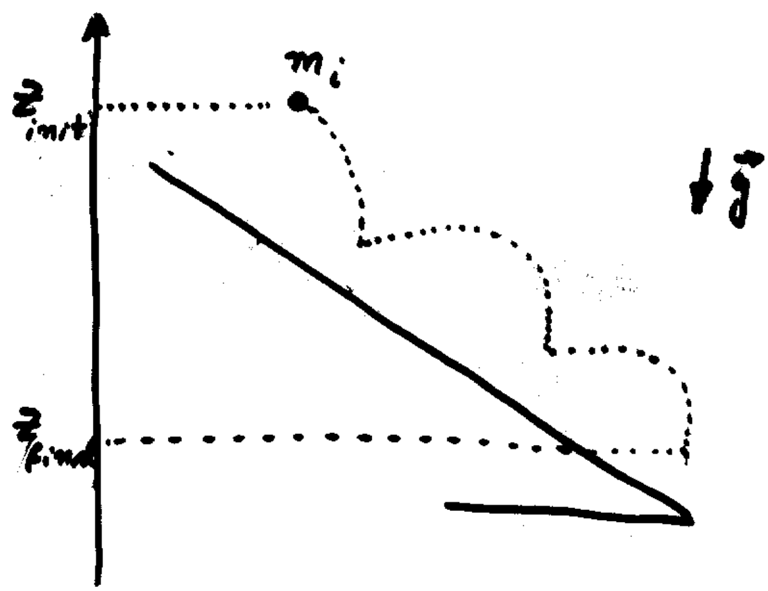
> The work done by the internal forces in a rigid body is ZERO.

Thus, the only force that produces work on the rolling block is the gravitational force.



We already know that regardless of the trajectory followed by the "small" particle m_i the work done by the gravitational force is

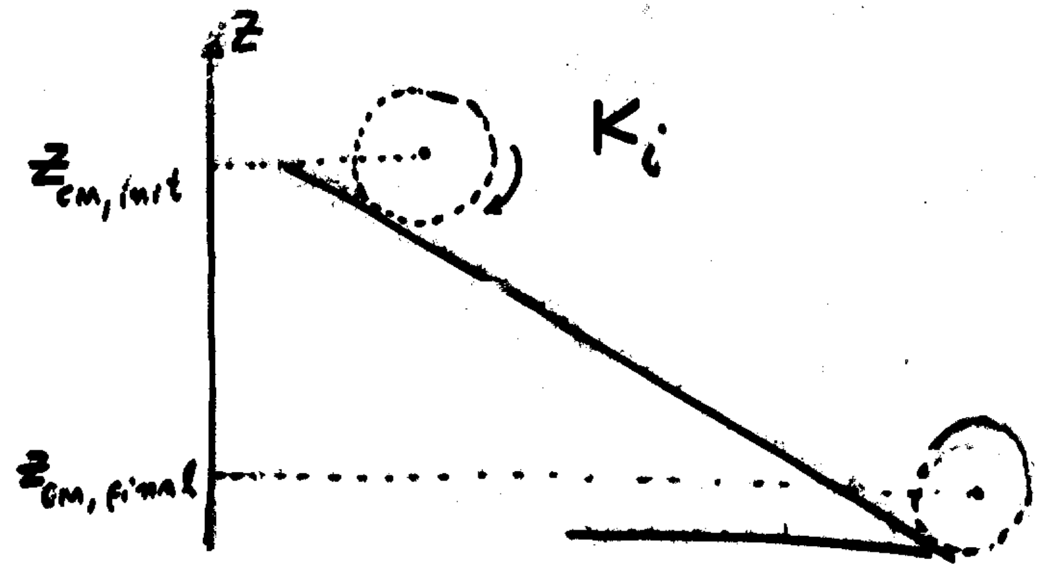
$$-m_i\, g\,(z_{final,i} - z_{init,i})$$



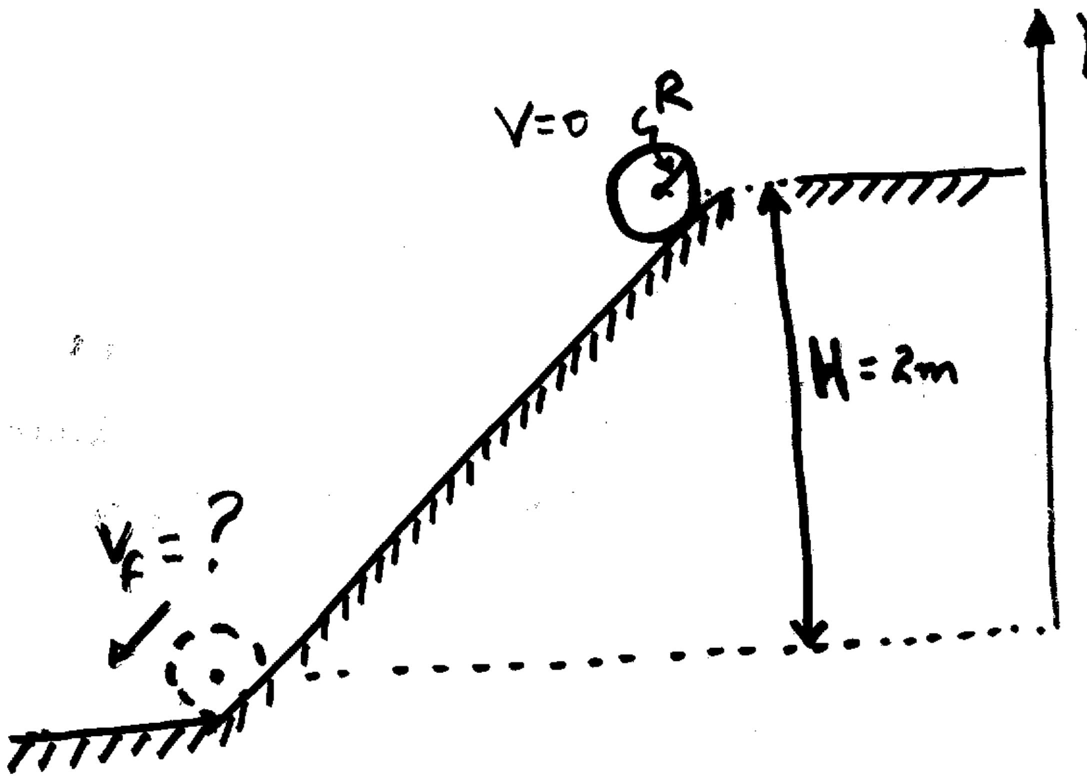
when we add all the work done on each particle, we obtain

$$W = -g\Big(\underbrace{\sum_i m_i z_{final,i}}_{M z_{cm,final}} - \underbrace{\sum_i m_i z_{init,i}}_{M z_{cm,initial}}\Big)$$

$$W = -\left(Mg\, z_{cm,final} - Mg\, z_{cm,initial}\right)$$



EXAMPLE A sphere, a cylinder, and a ring, all having the same radius R, roll down along a inclined plane, starting from rest at a height H. Find in each case the velocity of the CM when they arrive at the base of the plane.



$$E = \frac{1}{2} M V_{cm}^2 + \frac{1}{2} I \omega^2 + M g y_{cm}$$

Rigid body at the top

$$E = 0 + 0 + MgH$$

Rigid body at the bottom

$$E = \frac{1}{2} M V_f^2 + \frac{1}{2} I \omega_f^2 + Mg \cdot (0)$$

assuming NO slipping $\rightarrow \omega_f = \dfrac{V_f}{R}$

$$= \frac{1}{2} M V_f^2 + \frac{1}{2} I \frac{V_f^2}{R^2}$$

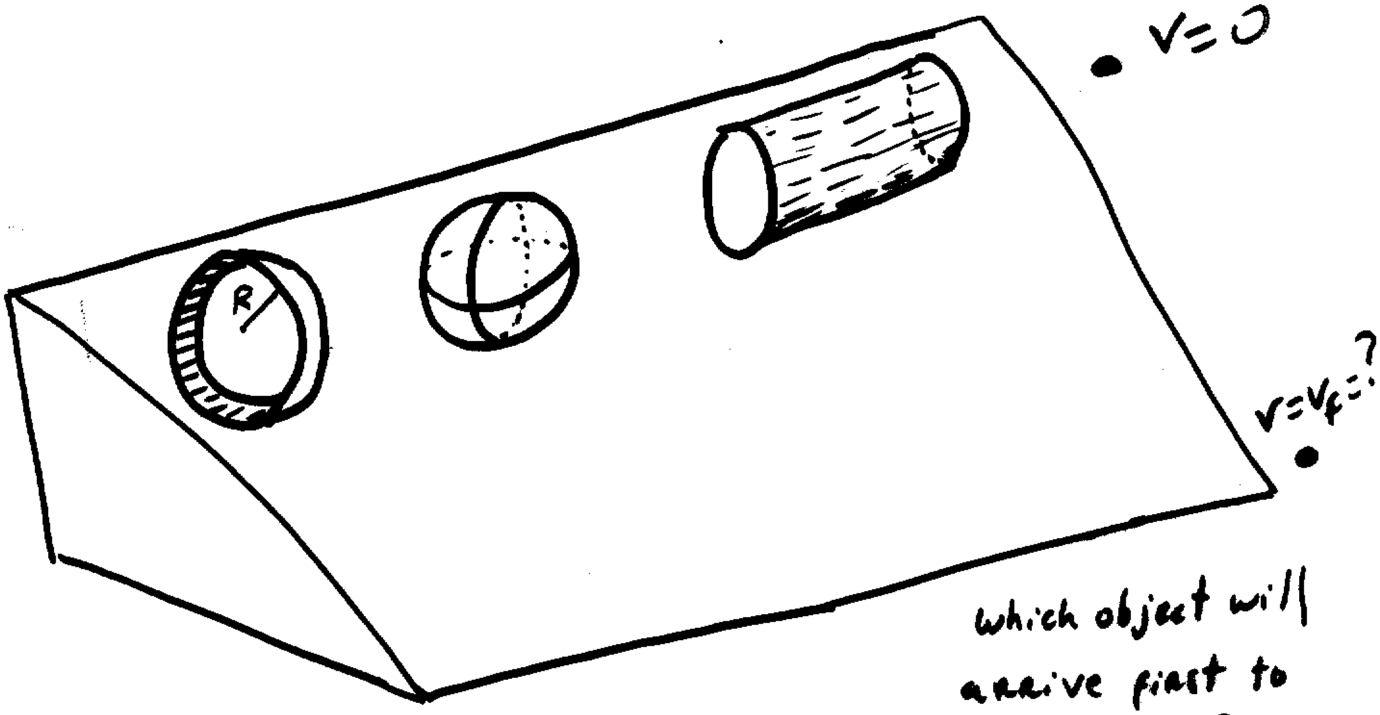
$$= \frac{1}{2}\left(M + \frac{I}{R^2}\right) V_f^2$$

Conservation of mechanical energy implies

$$\underbrace{\frac{1}{2}\left(M + \frac{I}{R^2}\right) V_f^2}_{\text{kinetic energy}} = \underbrace{MgH}_{\text{Amount of potential energy converted into}}$$

$$\frac{1}{2}\left(1 + \frac{I}{MR^2}\right) V_f^2 = gH$$

$$\boxed{V_f^2 = \frac{1}{1 + \frac{I}{MR^2}} \cdot 2gH}$$



which object will arrive first to the floor?

Ring $I = MR^2$ $\quad V_f^2 = gH$

Sphere $I = \frac{2}{5}MR^2$ $\quad V_f^2 = \frac{10}{7}gH = 1.43\,gH$

Cylinder $I = \frac{1}{2}MR^2$ $\quad V_f^2 = \frac{4}{3}gH = 1.33\,gH$

Notice: • V_f doesn't depend on R

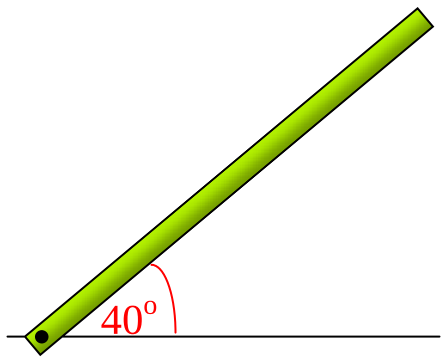
" " " M

• V_f depends only on the geometry

• Sphere arrives first to

EXAMPLE **Problem # 79, Ch-10 textbook**

A thin uniform rod of length L = 2.0 m and can pivot about a horizontal, frictionless pin through one end. It is released from rest at angle q = 40^o above the horizontal. Use the principle of conservation of mechanical energy to determine the angular speed of the rod as it passes through the horizontal position.



Solution