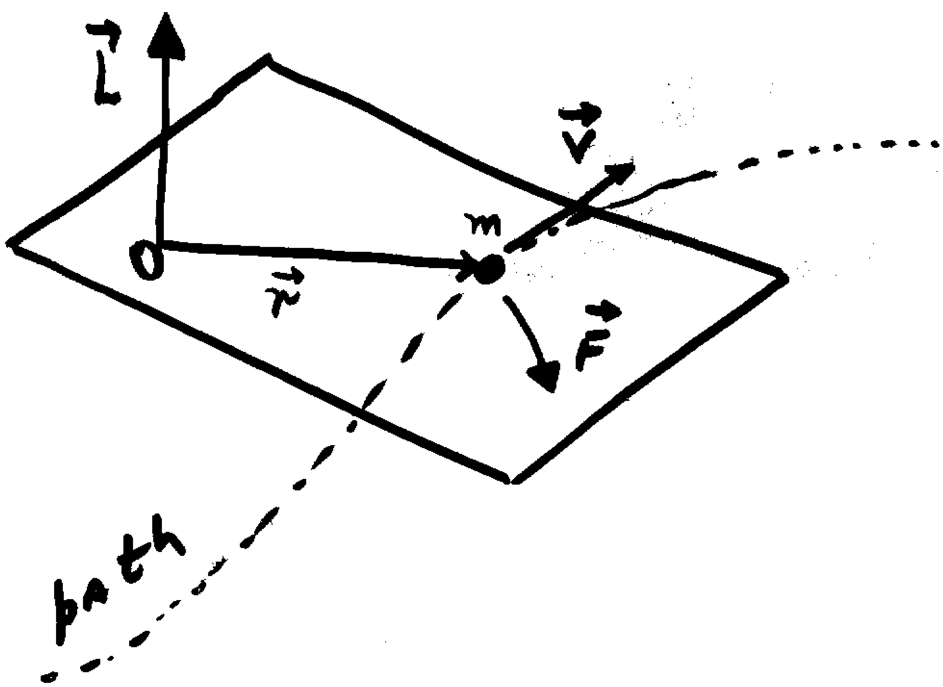
- $\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt} m\vec{v} = m\frac{d\vec{v}}{dt}$ translational motion

- $\vec{\tau} = \vec{r} \times \vec{F}$ causes rotation $\quad \vec{\tau} \stackrel{?}{=} \frac{d}{dt}(?)$

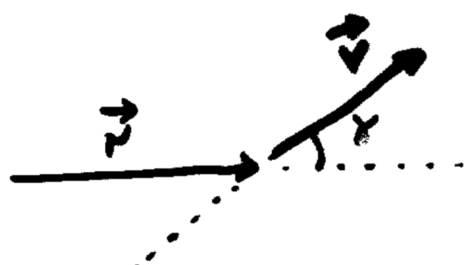
ANGULAR MOMENTUM $\vec{L}$

CASE: One particle



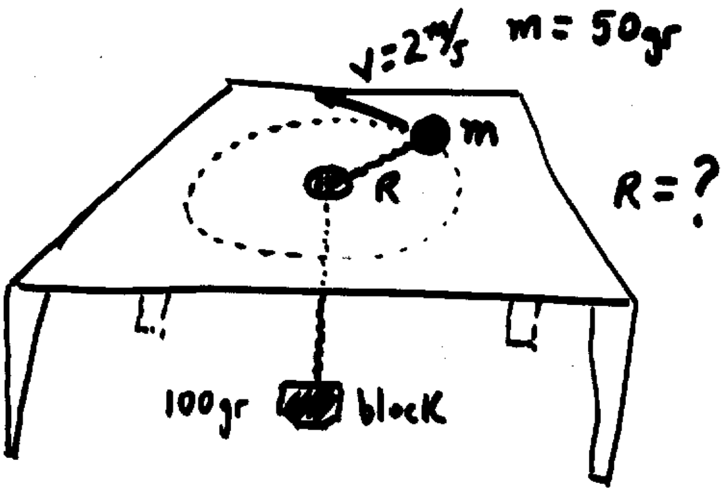
$\vec{L} = m\,\vec{r} \times \vec{v}$

$= \vec{r} \times \vec{p}$

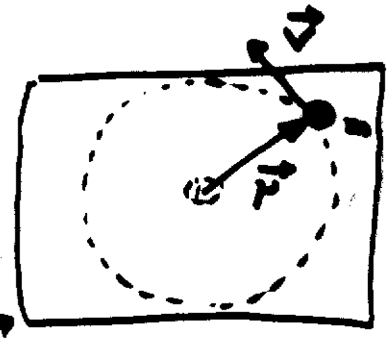


$|\vec{L}| = r p \sin\gamma$

Example Find the ANGULAR MOMENTUM of the of the mass "m".



LATERAL VIEW



$\vec{L} = m\, \vec{r} \times \vec{v}$

$L = m |\vec{r}| |\vec{v}| \sin 90°$

$= m R V$

✓ ? ✓

How to find R?

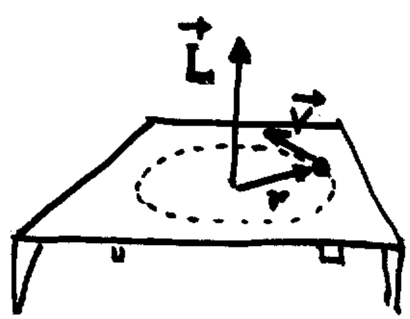
$$\underbrace{m \frac{v^2}{R}}_{\text{centripetal force needed for the circular motion}} = \underbrace{(100gr) \times g}_{\text{weight of the hanging block}}$$

$$\Rightarrow R = \frac{m\, v^2}{(100gr) \times g} = \frac{(50gr)\,(2\, m/s)^2}{(100gr) \times 9.8\, m/s^2} \approx 0.2 \text{ meter}$$

therefore:

$L = mRV = 0.05\, Kg \times 0.2\, m \times 2\, m/s$

$$\boxed{L = 0.02\ Kg\, m^2/s}$$



Relationship between $\vec{L}$ and $\vec{\tau}$

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\frac{d\vec{L}}{dt} = \vec{r} \times \frac{d}{dt}\vec{p} + \frac{d\vec{r}}{dt} \times \vec{p}$$

(but $\vec{p} = m\vec{v}$

$$= m\,\vec{r} \times \underbrace{\frac{d\vec{v}}{dt}}_{\vec{F}/m} + m\,\underbrace{\vec{v} \times \vec{v}}$$

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}$$

Thus $$\boxed{\frac{d\vec{L}}{dt} = \vec{\tau}}$$

- $\vec{L}$ and $\vec{\tau}$ are measured with respect to the same point
- If you apply a torque then you'll change $\vec{L}$

Compare this expression to $\frac{d\vec{P}}{dt} = \vec{F}$

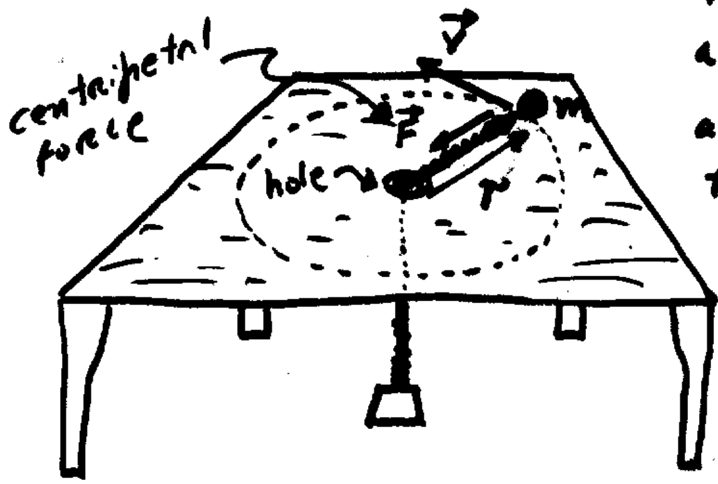
Notice: If $\vec{\tau} = \vec{0}$ then $\vec{L} = \text{const}$

Notice: $\vec{\tau} = \vec{r} \times \vec{F}$

Of course, if $\vec{F} = 0$ then $\vec{\tau} = 0$.

But, we can have some cases where $\vec{F} \neq 0$ and still $\vec{\tau} = 0$

For example:



mass "m", held by a string, circles around on a table

$\vec{F}$ is always pointing in the radial direction

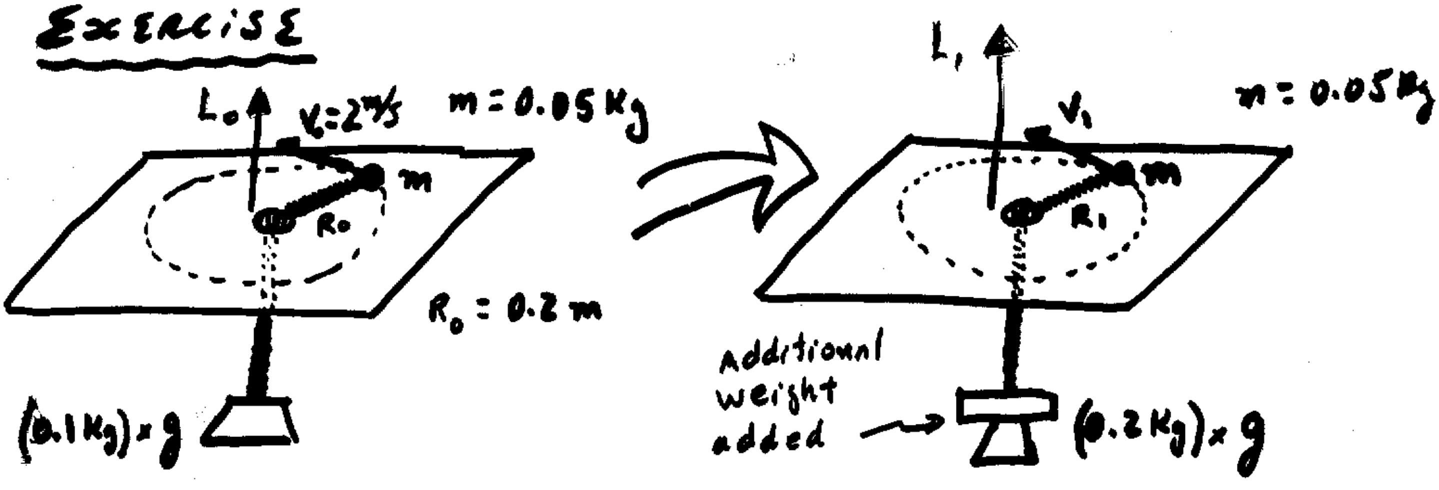
Tension $\rightarrow \vec{F} \parallel \vec{r}$ ($\vec{F}$ is parallel to $\vec{r}$)

Therefore: $\vec{F} \times \vec{r} = 0$

$$\vec{\tau} = 0$$

and $\vec{L} = \text{const}$

EXERCISE



$L_0 = m R_0 V_0$

$L_1 = ?$

$R_1 = ?$

$V_1 = ?$

The added weight increases the tension on the cord, that tension is responsible for the centripetal force $\vec{F}$ acting on the mass "m". Since the centripetal force $\vec{F}$ is parallel to vector position $\vec{r}$, $\vec{F} \times \vec{r} = 0$ and, therefore, the extra weight does not change the angular momentum. That is $L_1 = L_0$

$$L_0 = L_1 \Rightarrow m R_0 V_0 = m R_1 V_1 \quad \text{or} \quad \boxed{R_0 V_0 = R_1 V_1}$$

Also, $m \dfrac{V_1^2}{R_1} = (0.2\ \text{Kg})\, g$

$$\frac{(m R_1 V_1)^2}{m R_1^3} = (0.2\ \text{Kg})\, g \quad \text{or} \quad \frac{L_0^2}{m R_1^3} = (0.2\ \text{Kg})\, g \quad \boxed{R_1^3 = \frac{L_0^2}{m (0.2\ \text{Kg})\, g}}$$

$$R_1^3 = \frac{(2\times10^{-2})^2}{(5\times10^{-2})(0.2)\,9.8}\ m^3 \approx 4.1\times10^{-3}\ m^3$$

$$\boxed{R_1 = 0.16\ m}$$

$$V_1 = \frac{R_0 V_0}{R_1} = \frac{(0.2)(2)}{(0.16)}\ m/s \qquad \boxed{V_1 = 0.25\ m/s} \times 10$$

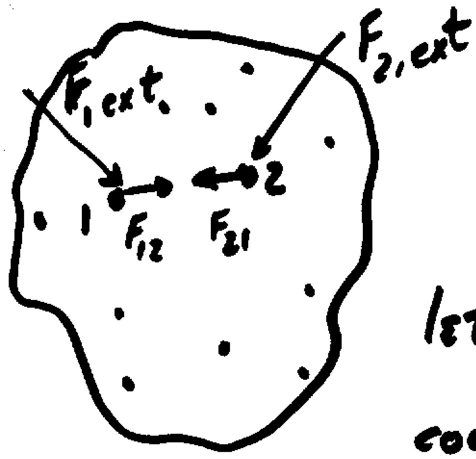
Conclusion:

the vector angular momentum $\vec{L}$ remained constant despite the fact that we increased the weight of the hanging block (see figure)

Eventhough the force acting on the mass "m" (which circles around) increased in magnitude, the torque of such force was ZERO. Therefore L remained constant

(we are applying $\frac{d\vec{L}}{dt} = \vec{\tau}$)

ANGULAR MOMENTUM
for a system of particles.



$$\boxed{\vec{L} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 + \dots}$$

(1)

let's focus our attention in a couple of particles (1 and 2 for example) and THEN generalize the result for all the other couple of particles

$$\frac{d}{dt}\vec{r}_1 \times \vec{p}_1 =$$

$$= \vec{r}_1 \times \frac{d\vec{p}_1}{dt} + \left(\frac{d\vec{r}_1}{dt}\right) \times \vec{p}_1$$

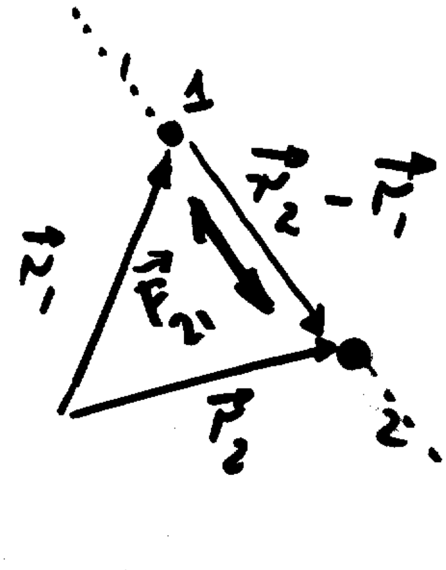
$$= \underbrace{\vec{r}_1 \times \vec{F}_1}_{\checkmark} + \underbrace{\vec{v}_1 \times m_1\vec{v}_1}_{=0}$$

$$\frac{d\vec{L}}{dt} = \vec{r}_1 \times \underbrace{\vec{F}_1}_{\vec{F}_{1\,ext} + \vec{F}_{12}} + \vec{r}_2 \times \underbrace{\vec{F}_2}_{\vec{F}_{2\,ext} + \vec{F}_{21}} + \dots$$

$$= \underbrace{\vec{r}_1 \times \vec{F}_{1\,ext}}_{\tau_{1\,ext}} + \underbrace{\vec{r}_2 \times \vec{F}_{2\,ext}}_{\tau_{2\,ext}} + \vec{r}_1 \times \underbrace{\vec{F}_{12}}_{-F_{21}} + \vec{r}_2 \times \vec{F}_{21} + \dots$$

$$= \vec{\tau}_{1\,ext} + \vec{\tau}_{2\,ext} + (\vec{r}_2 - \vec{r}_1) \times \vec{F}_{21} + \dots$$

$(\vec{r}_2 - \vec{r}_1)$ is a vector that is along the line that joins 1 and 2



Also $\vec{F}_{21}$ is along the line that joins 1 and 2

So, $(\vec{r}_2 - \vec{r}_1) \times \vec{F}_{21} =$

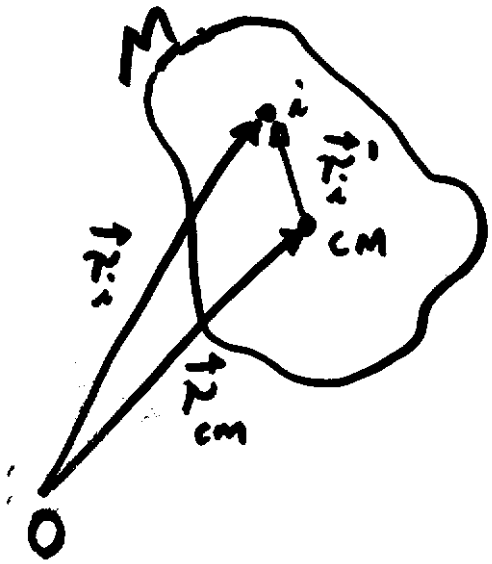
$$\frac{d\vec{L}}{dt} = \vec{\tau}_{1\,ext} + \vec{\tau}_{2\,ext} + \cdots$$

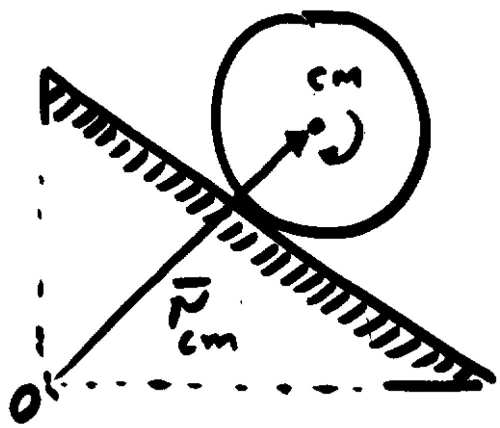
$$\boxed{\frac{d\vec{L}}{dt} = \vec{\tau}_{ext}}$$

ONLY EXTERNAL (FORCES) TORQUES cause $\vec{L}$ to change

2

Relationship between $\vec{L}$ and $\vec{L}_{cm}$





$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i = \sum_i m_i \vec{r}_i \times \vec{v}_i$$

$\vec{L}$ evaluated from "O"

$$\vec{L}_{cm} = \sum_i \vec{r}_i{}' \times \vec{p}_i{}' = \sum_i m_i \vec{r}_i{}' \times \vec{v}_i{}'$$

$\vec{L}_{cm}$ evaluated from the center of mass

Using
$$\vec{r}_i = \vec{r}_{cm} + \vec{r}_i{}'$$
$$\vec{v}_i = \vec{v}_{cm} + \vec{v}_i{}'$$

we obtain

$$\boxed{\vec{L} = \vec{L}_{cm} + M\, \vec{r}_{cm} \times \vec{v}_{cm}} \quad (3)$$

Relationship between $\vec{L}$ and $\vec{L}_{cm}$

$$\vec{L} = \sum_i m_i \vec{r}_i \times \vec{v}_i$$

$$\left(\vec{r}_i = \vec{r}_{cm} + \vec{r}_i\,' \quad \text{and} \quad \vec{v}_i = \vec{v}_{cm} + \vec{v}_i\,' \right) \Downarrow$$

$$= \sum_i m_i \left(\vec{r}_{cm} + \vec{r}_i\,'\right) \times \left(\vec{v}_{cm} + \vec{v}_i\,'\right)$$

$$= \sum_i \left(m_i \vec{r}_{cm} \times \vec{v}_{cm}\right) + \sum_i \left(m_i \vec{r}_{cm} \times \vec{v}_i\,'\right) + \sum_i \left(m_i \vec{r}_i\,' \times \vec{v}_{cm}\right) + \sum_i \left(m_i \vec{r}_i\,' \times \vec{v}_i\,'\right)$$

$$= \underbrace{\left(\sum_i m_i\right)}_{M} \vec{r}_{cm} \times \vec{v}_{cm} + \vec{r}_{cm} \times \underbrace{\left(\sum_i m_i \vec{v}_i\,'\right)}_{\substack{\text{velocity of the center of mass}\\ \text{with respect to the center of mass}\\ \text{Therefore this term is zero}}} + \underbrace{\left(\sum_i m_i \vec{r}_i\,'\right)}_{\substack{\text{position of the center of mass}\\ \text{with respect to the center of mass.}\\ \text{Therefore this term is zero}}} \times \vec{v}_{cm} + \underbrace{\sum_i m_i \vec{r}_i\,' \times \vec{v}_i\,'}_{L_{cm}}$$

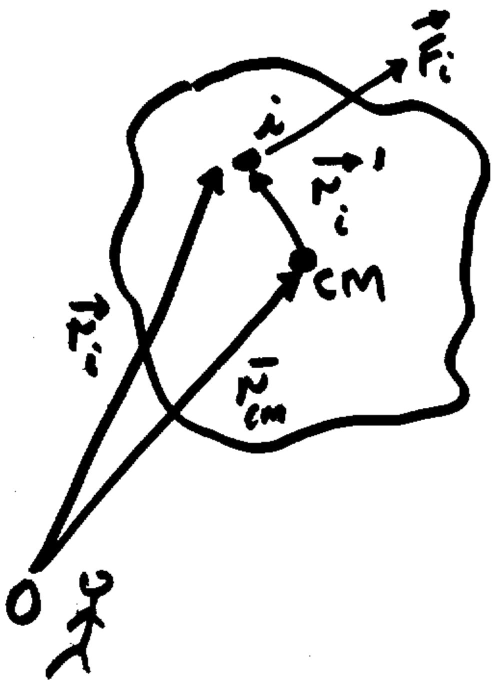
$$\boxed{\vec{L} = M \vec{r}_{cm} \times \vec{v}_{cm} + \vec{L}_{cm}}$$

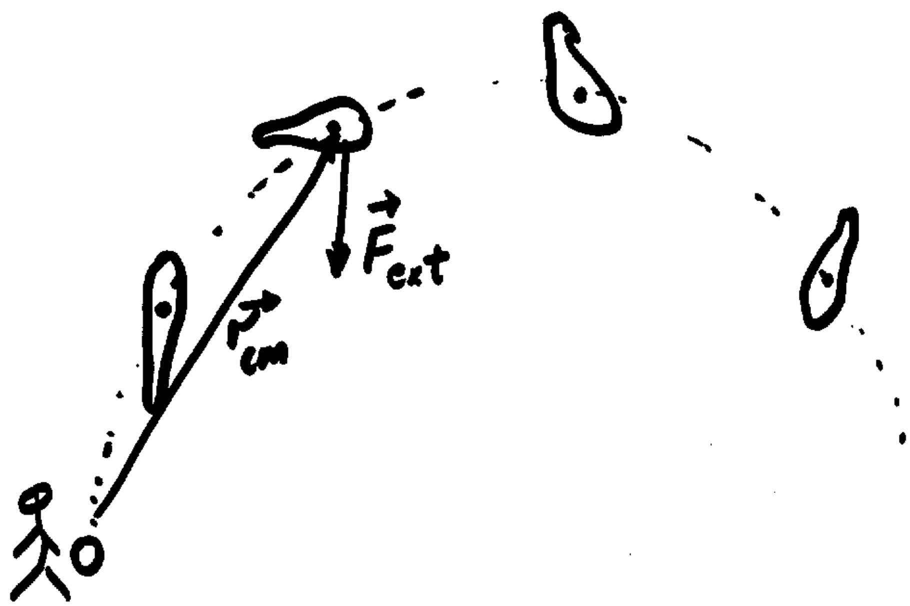
If you are not completely convinced, notice this

$$\sum_i m_i \vec{v}_i\,' = \sum_i m_i \left(\vec{v}_i - \vec{v}_{cm}\right) = \underbrace{\sum_i m_i \vec{v}_i}_{M\vec{v}_{cm}} - \sum_i m_i \vec{v}_{cm} = M v_{cm} - \underbrace{\sum_i m_i}_{M} \vec{v}_{cm}$$

$$= M\vec{v}_{cm} - M\vec{v}_{cm} = 0$$

Relationship between $\vec{\tau}$ and $\vec{\tau}_{cm}$





$$\vec{\tau}_{ext} = \sum_i \vec{r}_i \times \vec{F}_{i\,ext}$$

$\vec{\tau}$ is the external torque evaluated from "O"

$$\vec{\tau}_{cm} = \sum_i \vec{r}_i\,' \times \vec{F}_{i\,ext}$$

$\vec{\tau}_{cm}$ is the external torque evaluated from the center of mass

Using $\vec{r}_i = \vec{r}_{cm} + \vec{r}_i\,'$ we obtain

$$\boxed{\vec{\tau}_{ext} = \vec{\tau}_{cm} + \vec{r}_{cm} \times \vec{F}_{ext}} \quad (4)$$

Notice: A reference attached to the center of mass (CM) is not a inertial system.

Despite this, we can obtain the following:

Using (3)

$$\frac{d\vec{L}}{dt} = \frac{d\vec{L}_{cm}}{dt} + M\,\vec{r}_{cm}\times\underbrace{\frac{d\vec{V}_{cm}}{dt}}_{\vec{a}_{cm}} + M\,\frac{d\vec{r}_{cm}}{dt}\times\vec{V}_{cm}$$

$$= \frac{d}{dt}\vec{L}_{cm} + \vec{r}_{cm}\times\underbrace{M\,\vec{a}_{cm}}_{\vec{F}_{ext}}$$

$$\frac{d}{dt}\vec{L} = \frac{d}{dt}\vec{L}_{cm} + \vec{r}_{cm}\times\vec{F}_{ext}$$

$\downarrow$ using (2)

$$\boxed{\vec{\tau}_{ext} = \frac{d}{dt}\vec{L}_{cm} + \vec{r}_{cm}\times\vec{F}_{ext}} \quad (5)$$

Comparing 4 and 5 we conclude

$$\boxed{\vec{\tau}_{cm} = \frac{d}{dt}\vec{L}_{cm}} \qquad (6)$$

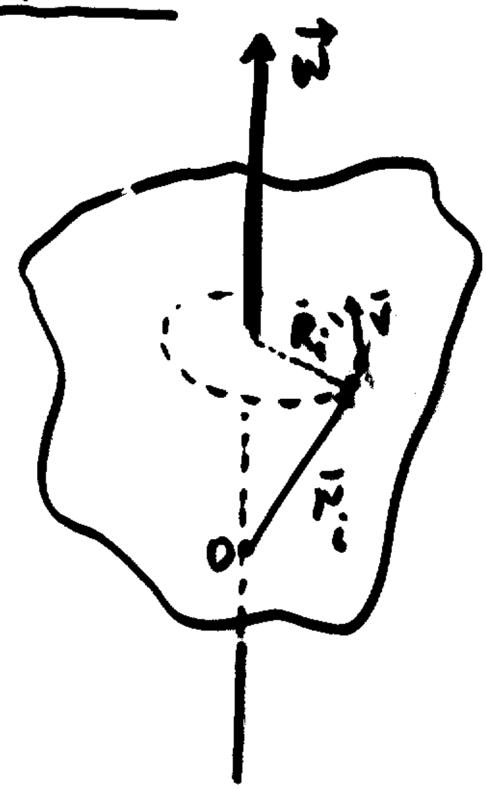
- everything is evaluated from the center of mass
- this expression is valid, despite the fact that the center of mass may be accelerated !

What we have so far is:

$$\left.\begin{aligned}\vec{L} &= \sum_i \vec{r}_i \times \vec{p}_i \\ \vec{\tau}_{ext} &= \sum_i \vec{r}_i \times \vec{F}_{i\,ext}\end{aligned}\right\}\quad \frac{d\vec{L}}{dt} = \vec{\tau}_{ext}$$

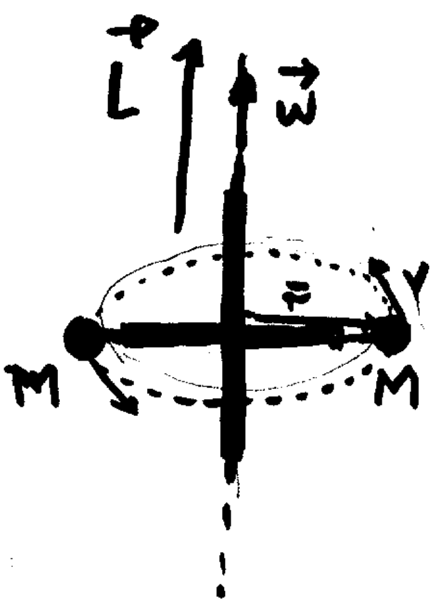
$$\left.\begin{aligned}\vec{L} &= \vec{L}_{cm} + M\,\vec{r}_{cm} \times \vec{V}_{cm} \\ &= \vec{L}_{cm} + \vec{r}_{cm} \times \vec{P}_{cm} \\ \vec{\tau}_{ext} &= \vec{\tau}_{cm} + \vec{r}_{cm} \times \vec{F}_{ext}\end{aligned}\right\}\quad \frac{d\vec{L}_{cm}}{dt} = \vec{\tau}_{cm}$$

Angular Momentum of a Rigid Body

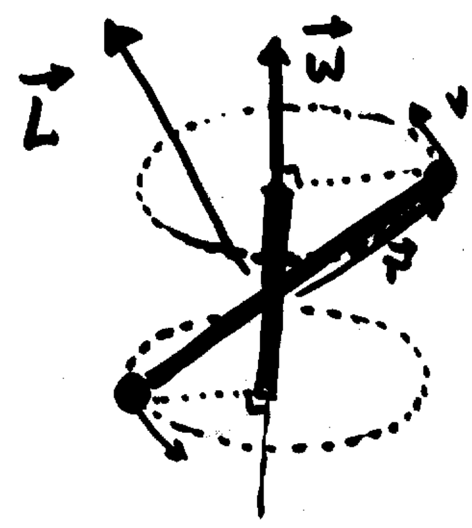


$$\vec{L} = \sum_i m_i \vec{r}_i \times \vec{v}_i$$

RIGID BODY:
the distance between its component particles remain fixed under the application of a force or torque. $\longrightarrow$ conserves its shape during its motion



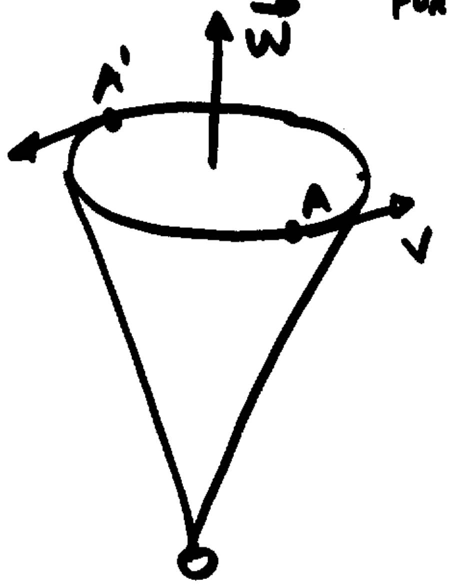
$\vec{L} \parallel \vec{w}$

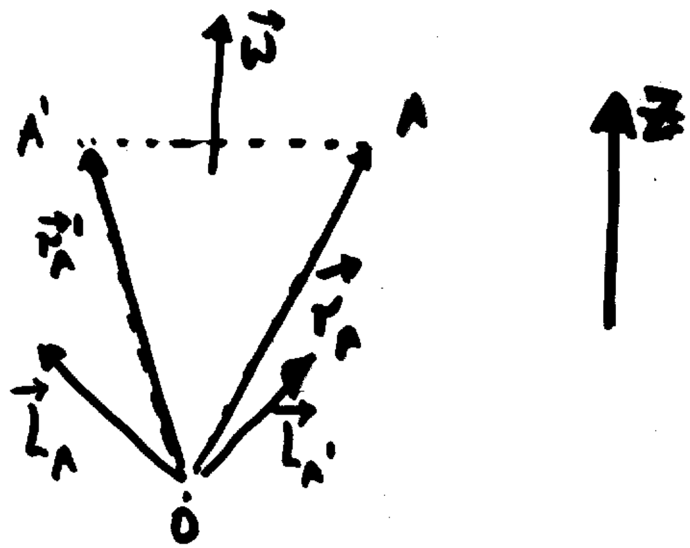


- In general $\vec{L}$ is <u>not</u> parallel to $\vec{w}$

- But, in some cases $\vec{L}$ and $\vec{w}$ are parallel.

Let's consider a symmetric rigid body: for example





The contributions from A and A' to the angular momentum results in a net Z component. (the horizontal components of $\vec{L}_A$ and $\vec{L}_A$' cancell out

So, for symmetric objects, rotating around one axis of symmetry: $\vec{L}$ is parallel to $\vec{w}$

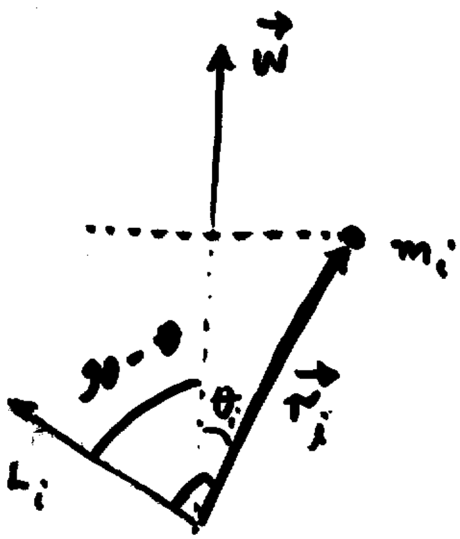
- What we are saying is:
  - In general $\vec{L}$ and $\vec{\omega}$ are <u>not</u> parallel
  - For symmetric looking objects, it is easier to identify their symmetry axis. And, if we make such objects to turn around one of its axis of symmetry then it turns out that $\vec{L}$ is parallel to $\vec{\omega}$

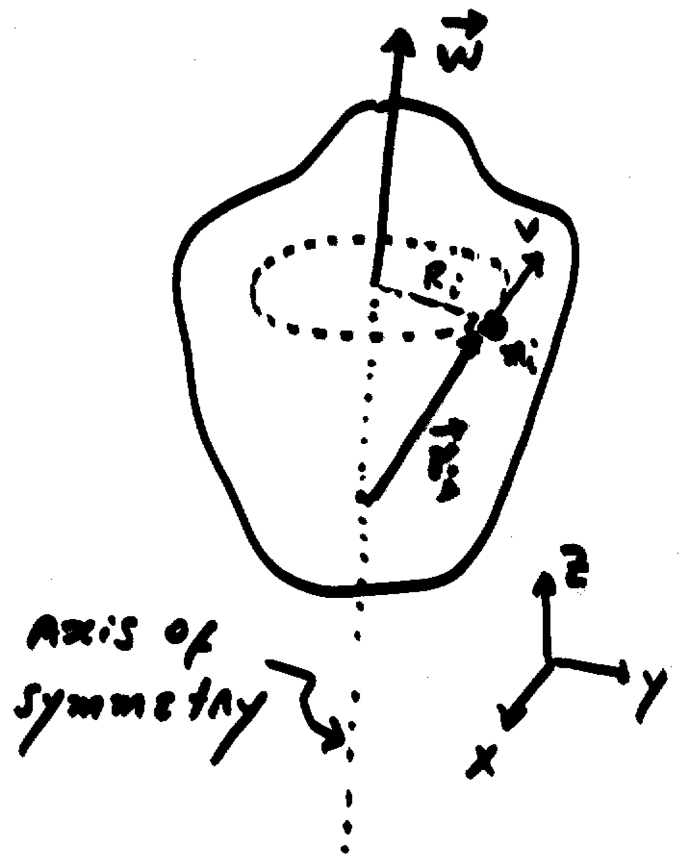
Having said that, let me emphasize the following:

- Even deformed-looking rigid-bodies have axis of symmetry. It is just harder to identify them.

  If we make a deformed-looking rigid-body rotate around one of its axis of symmetry then it turns out that $\vec{L}$ will be parallel to $\vec{\omega}$

Let's consider a rigid body (either a deformed-looking or a symmetric-looking one] is rotating around one of its axis of symmetry. And, let's call that axis the z-axis.





Convince yourself that the vectors $\vec{r}_i$ and $\vec{v}_i$ always make $90°$. Therefore

$$\vec{L}_i = m_i \vec{r}_i \times \vec{v}_i \text{, where}$$

$$L_i = m_i r_i v_i \sin 90° = m_i r_i v_i \qquad (7)$$

Since the rigid body is rotating around its symmetry axis (Z-Axis), we are interested only in the z-component of $\vec{L}_i$. (The horizontal components $\underbrace{\text{will}}_{\text{of } \vec{L}_i}$ cancel out; we know $\vec{L} = \sum \vec{L}_i$ will be parallel to $\vec{\omega}$, this is parallel to the z-Axis)

Thus $$\left(\vec{L}_i\right)_z = L_i \cos(90-\theta)$$
$$= L_i \sin\theta_i$$
$$= m_i v_i \underbrace{r_i \sin\theta_i}$$

Notice from figure that $r_i \sin\theta_i = R_i$ where R_i is the distance from m_i to the rotation axis

$$\left(\vec{L}_i\right)_z = m_i R_i \underbrace{v_i}$$

$v_i = R_i \omega$

The linear velocity V_i is related to the angular velocity w through the expression $V_i = R_i w$. Using this expression in our previous formula we obtain

$$(\vec{L}_i)_z = m_i R_i^2 w$$

When we add the contribution from all the particles that make up the rigid body we obtain

$$\boxed{L_z = \underbrace{\left(\sum_i m_i R_i^2\right)}_{} w} \qquad (8)$$

geometric factor associated with a given rigid-body

It is called MOMENTUM OF INERTIA I