So, what is a vector ?

Is (2, 3, 17) a vector ?

Just by itself (without providing any more information) this question is meaningless.

But, if we are told that these three numbers are always to be calculated as the component (along the axis of a system of coordinates) of a certain physical quantity (velocity, position, or force, for example), then we are in a better position to give an answer.

Notice, we are still ellusive to give you an answer (YES or no) for the question above.

What happens is that no any set of three numbers, (2, 3, 17) for example, is the representation of a vector.

the concept of vector is associated to the transformation of coordinates

IF the components (V_1, V_2, V_3), along the axis of a coordinate system, of a given physical property

transform to (V_1', V_2', V_3'), when using another coordinate system, according to some given rules

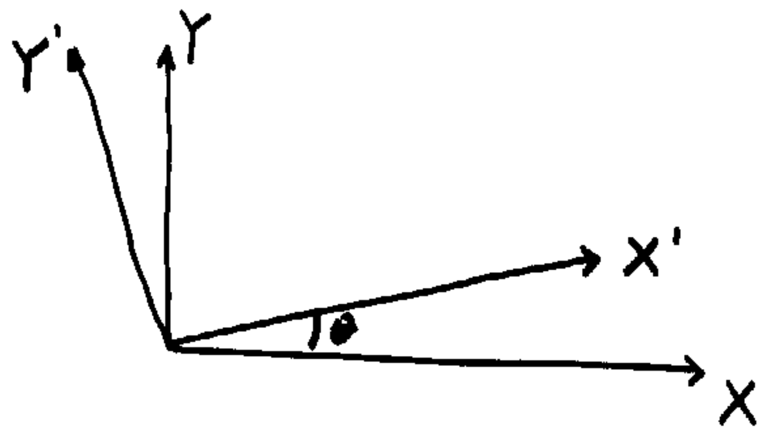
THEN

we say that that physical property is a vector.

What type of rules are we referring to?

Answer: those established by the transformation of coordinates.

Transformation of coordinates



For simplicity, let's consider just a 2-dimensional rotation

$$\hat{i}' = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} + 0\,\hat{k}$$
$$\hat{j}' = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} + 0\,\hat{k} \qquad (1)$$
$$\hat{k}' = 0\,\hat{i} \quad 0\,\hat{j} + 1\,\hat{k}$$

Also

$$\hat{i} = \cos\theta\,\hat{i}' - \sin\theta\,\hat{j}' + 0\,\hat{k}'$$
$$\hat{j} = +\sin\theta\,\hat{i}' + \cos\theta\,\hat{j}' + 0\,\hat{k}' \qquad (2)$$
$$\hat{k} = 0\,\hat{i}' + 0\,\hat{j}' + 1\,\hat{k}'$$

Let $\vec{F}$ be a physical quantity, such that

$$\vec{F} = F_x\,\hat{i} + F_y\,\hat{j} + F_z\,\hat{k}$$

Using (2) we obtain

$$\vec{F} = F_x\left(\cos\theta\,\hat{i}' - \sin\theta\,\hat{j}'\right) +$$

$$F_y\left(\sin\theta\,\hat{i}' + \cos\theta\,\hat{j}'\right) +$$

$$F_z\,\hat{k}'$$

$$= \left(\cos\theta\,F_x + \sin\theta\,F_y + 0\,F_z\right)\hat{i}' +$$

$$\left(-\sin\theta\,F_x + \cos\theta\,F_y + 0\,F_z\right)\hat{j}' +$$

$$\left(\;0\,F_x \qquad 0\,F_y + F_z\right)\hat{k}'$$

Thus,

$$\vec{F} = F_x\,\hat{i} + F_y\,\hat{j} + F_z\,\hat{k} = F_x'\,\hat{i} + F_y'\,\hat{j} + F_z'\,\hat{k}$$

implies:

$$\begin{pmatrix} F_x' \\ F_y' \\ F_z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} \qquad (3)$$

We give now the definition of a VECTORIAL physical quantity:

If the components of a physical quantity Q transform according to (3) (when using two different coordinate system) then Q is a VECTOR let $\bar{a} = (a_1, a_2, a_3)$

$\bar{b} = (b_1, b_2, b_3)$

be vectors

Is $\bar{c} = \frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}$ a vector?

ANSWER

$$\Big(\underbrace{\cos\theta\, a_1 + \sin\theta\, a_2}_{a_1'}, \underbrace{-\sin\theta\, a_1 + \cos\theta\, a_2}_{a_2'}, \underbrace{a_3}_{a_3'}\Big) = \vec{a}$$

$$\Big(\underbrace{\cos\theta\, b_1 + \sin\theta\, b_2}_{b_1'}, \underbrace{-\sin\theta\, b_1 + \cos\theta\, b_2}_{b_2'}, \underbrace{b_3}_{b_3'}\Big) = \vec{b}$$

$$\bar{c} = \left(\overbrace{\frac{\cos\theta\, a_1 + \sin\theta\, a_2}{\cos\theta\, b_1 + \sin\theta\, b_2}}^{c_1'}, \overbrace{\frac{-\sin\theta\, a_1 + \cos\theta\, a_2}{-\sin\theta\, b_1 + \cos\theta\, b_2}}^{c_2'}, \overbrace{\frac{a_3}{b_3}}^{c_3'} \right)$$

Is
$$\begin{pmatrix} c_1' \\ c_2' \\ c_3' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{a_1}{b_1} \\ \frac{a_2}{b_2} \\ a_3/b_3 \end{pmatrix} ?$$

i.e. Is $\frac{\cos\theta\, a_1 + \sin\theta\, a_2}{\cos\theta\, b_1 + \sin\theta\, b_2} = \frac{a_1}{b_1}\cos\theta + \frac{a_2}{b_2}\sin\theta$?

NO.

Therefore $\bar{c}$ is not a vector