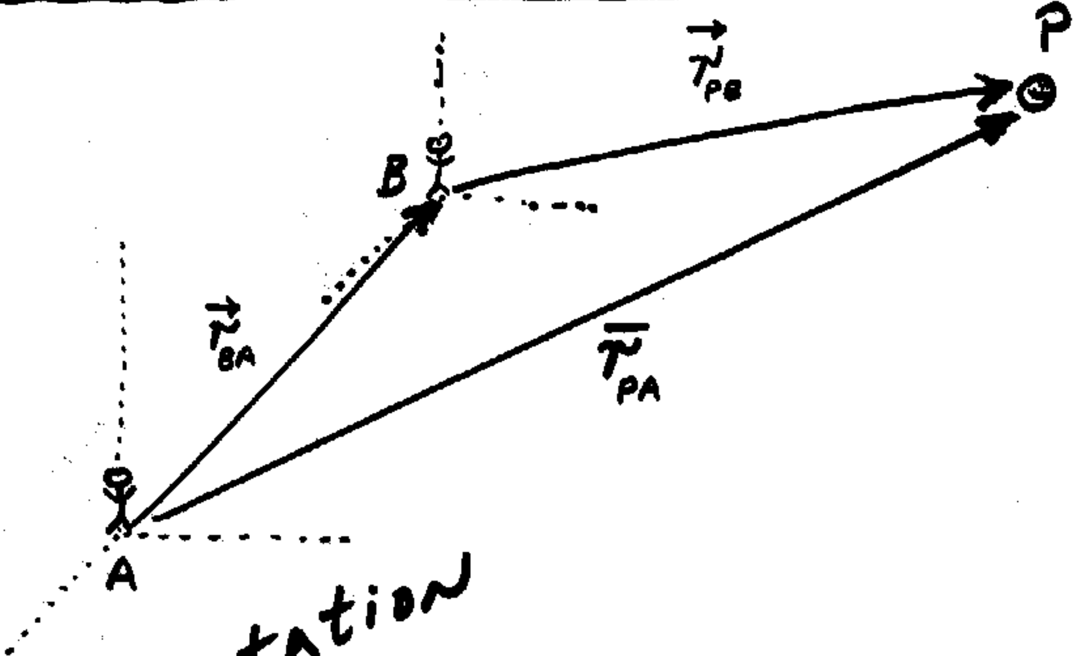
MOTION in TWO-DIMENSIONS



notation

$\vec{r}_{BA}$ position of B with respect to A

$\vec{r}_{PA}$ position of P with respect to A

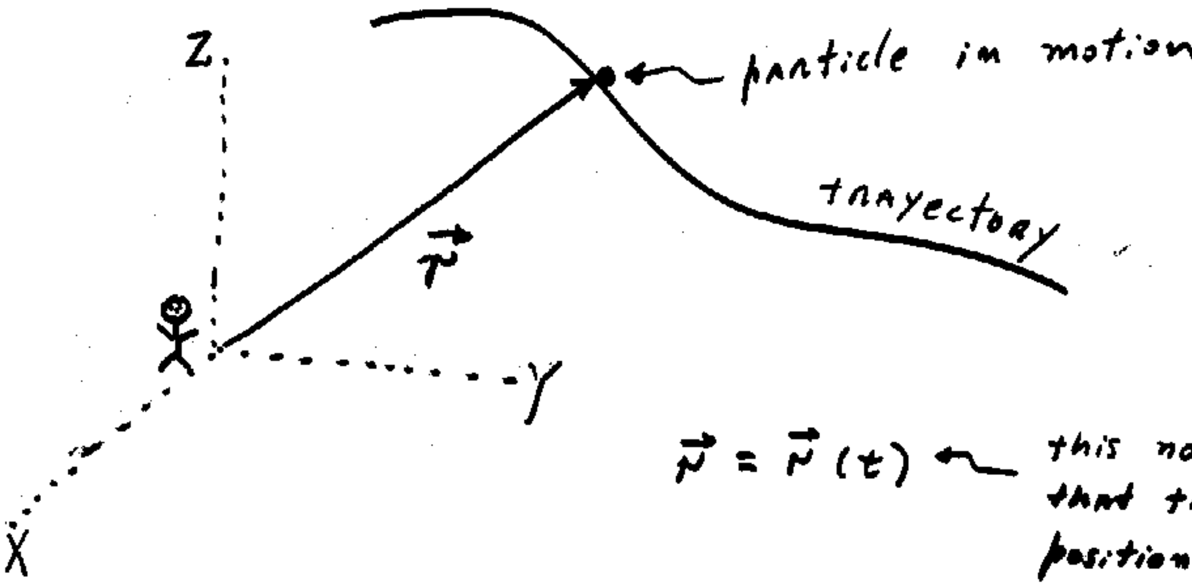
$\vec{r}_{PB}$ position of P with respect to B

$$\boxed{\vec{r}_{BA} + \vec{r}_{PB} = \vec{r}_{PA}}$$

or

$$\boxed{\vec{r}_{PB} = \vec{r}_{PA} - \vec{r}_{BA}}$$

VECTOR POSITION

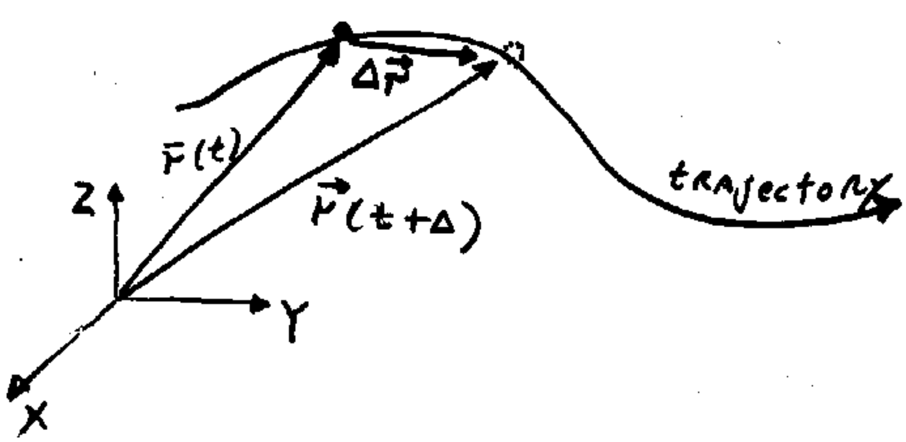


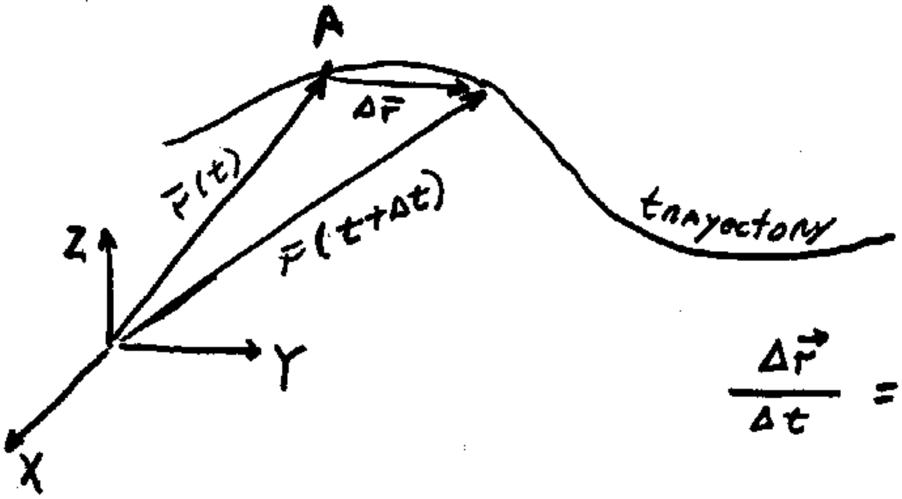
$\vec{r} = \vec{r}\,(t)$ ← this notation means that the vector position $\vec{r}$ is a function of time

With respect to a system of reference:

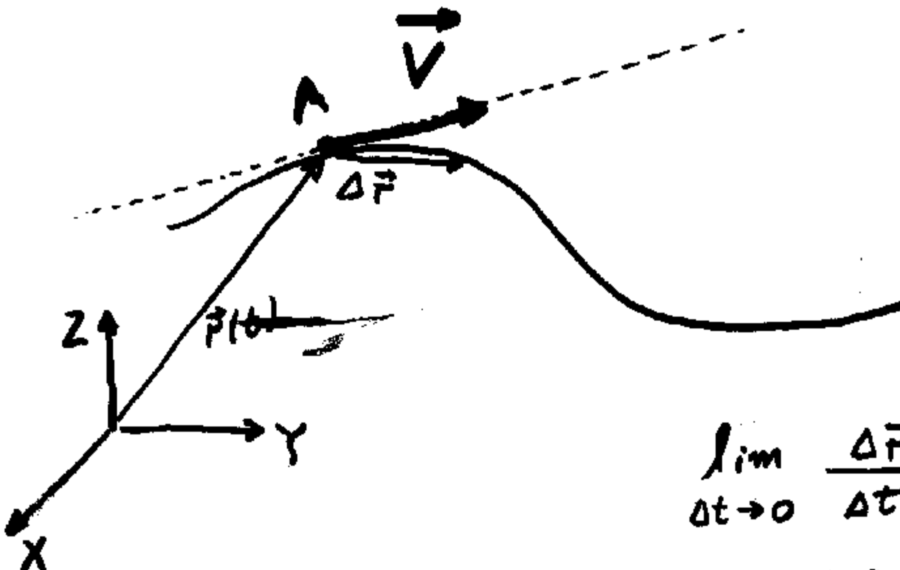
$$\boxed{\vec{r}\,(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}}$$

VECTOR DISPLACEMENT $\Delta\vec{r}$





$$\frac{\Delta\vec{r}}{\Delta t} = \,? \quad \text{interpretation?}$$



$$\lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \vec{V} \quad \text{instantaneous velocity}$$

$$\frac{d\vec{r}}{dt} = \vec{V}$$

<u>VELOCITY</u>

$$\vec{V} = \frac{d}{dt}\vec{r}(t)$$

$$= \frac{d}{dt}\left(x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k} \right)$$

$$= \frac{d\,x(t)}{dt}\,\hat{i} + \frac{d}{dt}y(t)\,\hat{j} + \frac{d}{dt}z(t)\,\hat{k}$$

$$= V_x\,\hat{i} + V_y\,\hat{j} + V_z\,\hat{k}$$

ACCELERATION

$$\vec{a} = \frac{d\vec{V}}{dt}$$

$$\bar{a} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$$

$$\vec{a} = \frac{d}{dt}\vec{V}$$

$$= \underbrace{\frac{d}{dt}\overbrace{\frac{d\vec{r}}{dt}}}= \frac{d^2}{dt^2}\bar{r}$$

$$\bar{a} = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k} \quad \text{def}$$

valid expression for <u>any</u> kind of motion

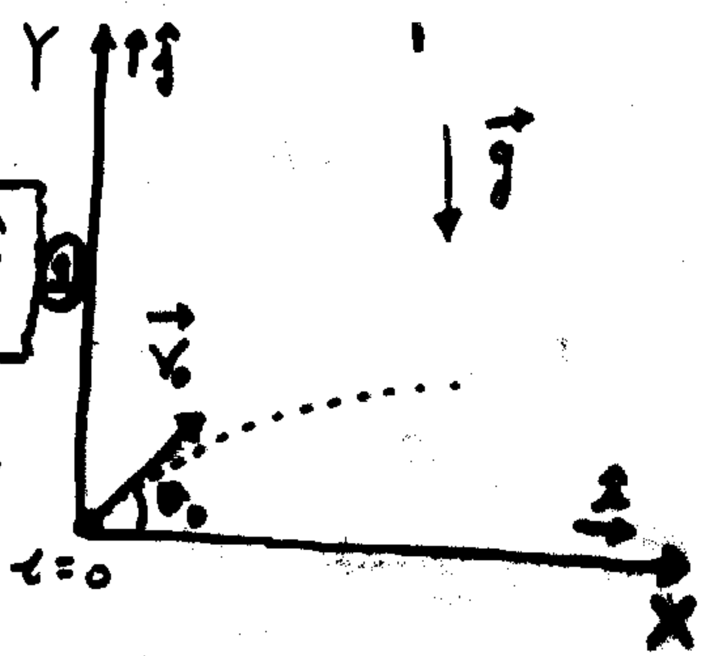
PROJECTILE MOTION

Motion (in 2D or 3D) of a particle near the surface of the Earth. $\Leftrightarrow$ ACCELERATION = const = $\vec{g}$

- In general

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$\boxed{\vec{a} = \frac{dV_x}{dt}\hat{i} + \frac{dV_y}{dt}\hat{j} + \frac{dV_z}{dt}\hat{k}} \quad 1$$



In the particular case of projectil motion, we know that:

$$\vec{a} = \vec{g} = g_n(-\hat{j})$$

$$\boxed{\vec{a} = -g\hat{j}} \quad 2$$

$V_{0x} = ?$

$V_{0y} = ?$

where $g = 9.81\ m/s^2$

$\vec{V}_0$: initial velocity

On one hand:

$$\vec{a} = \frac{dV_x}{dt}\,\hat{i} + \frac{dV_y}{dt}\,\hat{j} + \frac{dV_z}{dt}\,\hat{k}$$

on the other hand

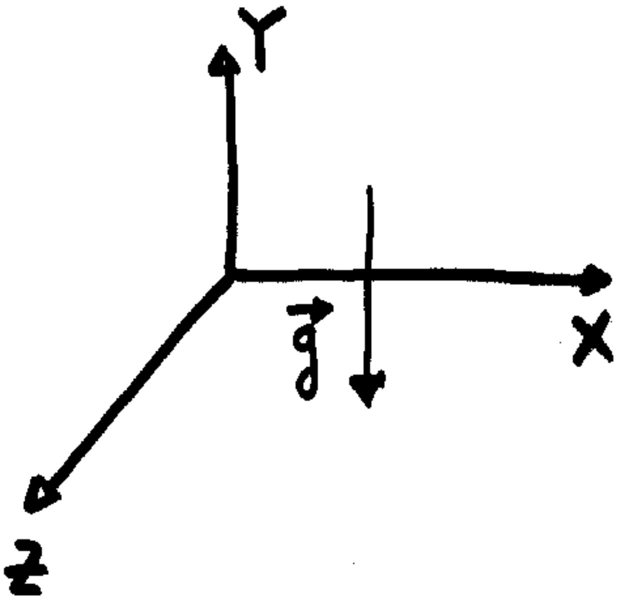
$$\vec{a} = 0\,\hat{i} - g\,\hat{j} + 0\,\hat{k}$$

If these two vectors ① and ② are to be equal, we must have:

$$\frac{dv_x}{dt} = 0$$

$$\frac{dv_y}{dt} = -g$$

$$\frac{dv_z}{dt} = 0$$



Further

$$\frac{dv_x}{dt} = 0 \Rightarrow \boxed{v_x = \text{const} = v_{ox}}$$

$$\frac{dv_y}{dt} = -g \Rightarrow \boxed{v_y = v_{oy} - g\,t}$$

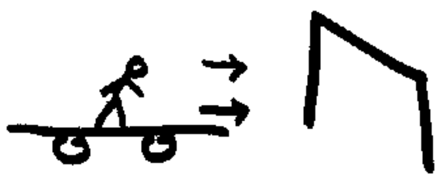
③

Notice: $\vec{g}$ affects only the vertical motion!

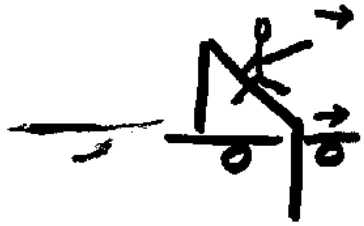
Consequences:

HORIZONTAL MOTION

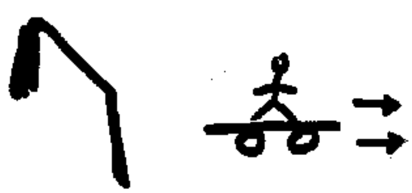
- $V_x = \text{Const} \equiv V_{0x}$

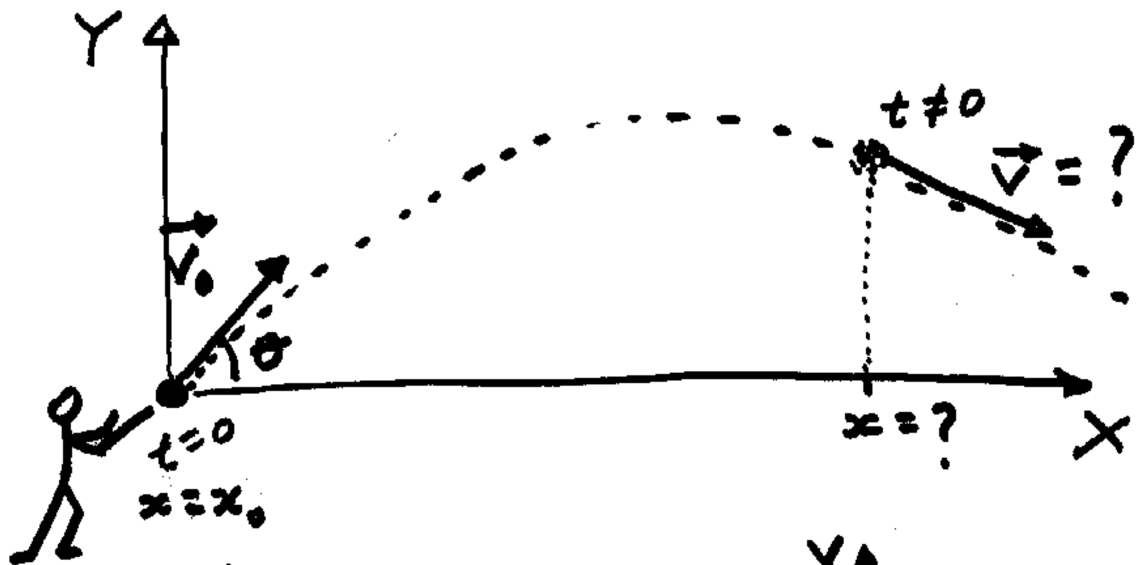


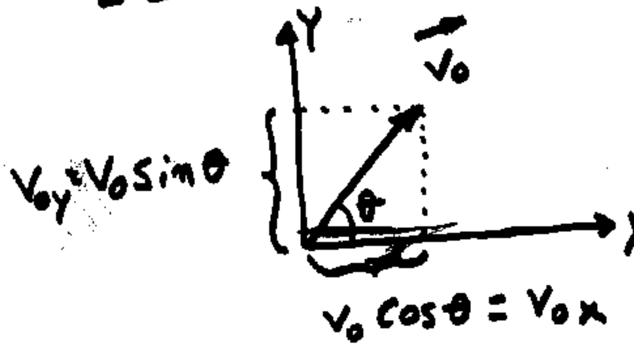
the rider jumps vertically



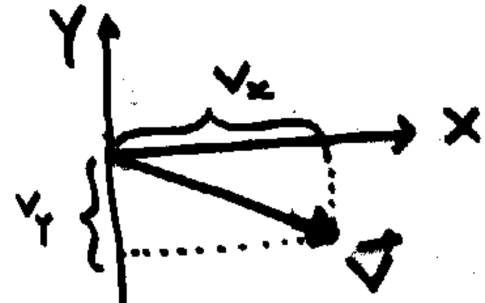
vector $\rightarrow$ and $\rightarrow$ are equal all the time.







$t = 0$



$t \neq 0$

From expression (3) we know:

$$\boxed{V_x = V_{0x}}$$

In general we know:

$$V_x = \frac{dx}{dt}$$

and, for this case, we have

$$\frac{dx}{dt} = V_{0x} \leftarrow \text{const value}$$

therefore

Description of the horizontal motion →

$$\boxed{\begin{aligned} x &= x_0 + V_{0x} t \\ &= x_0 + (V_0 \cos\theta) t \end{aligned}}$$

THE VERTICAL MOTION

From expression (3) we obtain:

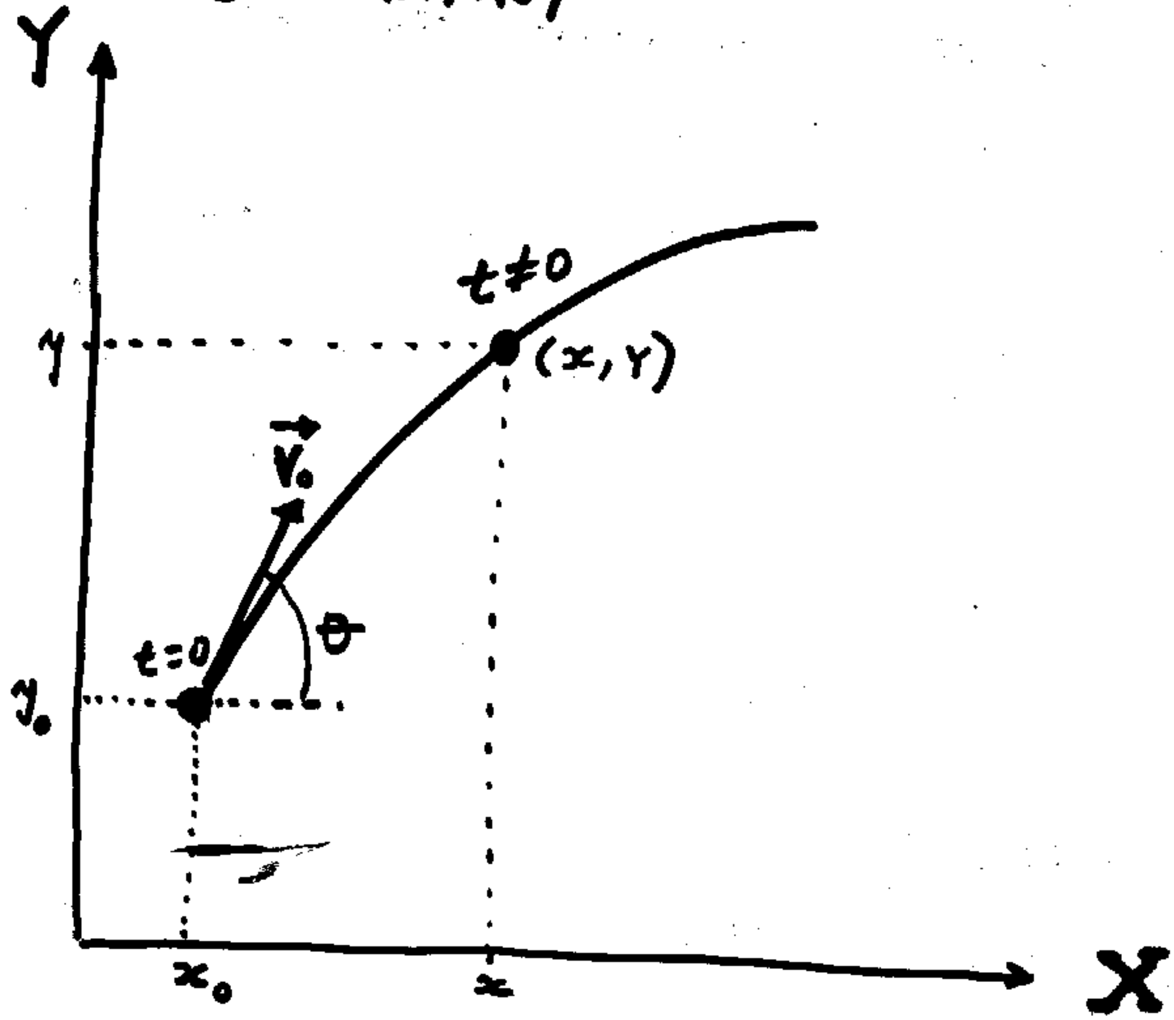
- $\frac{dV_y}{dt} = -g \Rightarrow V_y = V_{0Y} - gt$

$g = 9.8 \frac{m}{s^2}$ $\boxed{V_Y = V_0 \sin\theta_0 - gt}$

- $\frac{dY}{dt} = V_Y \Rightarrow \frac{dY}{dt} = V_0 \sin\theta_0 - gt$

$\Rightarrow \boxed{Y - Y_0 = (V_0 \sin\theta_0)t - \frac{1}{2}g t^2}$

SUMMARY



$$x = x_0 + (v_0 \cos\theta)\, t$$

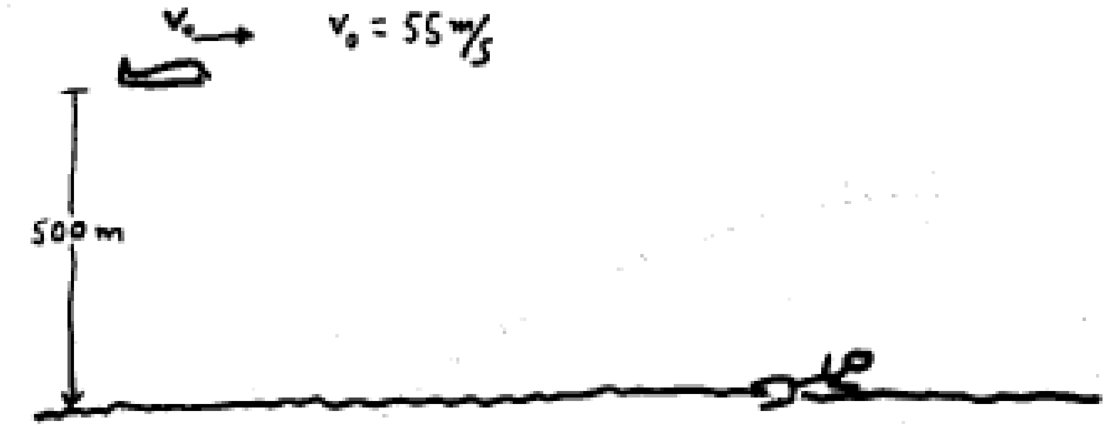
$v_x = v_0 \cos\theta$ (const at all time)

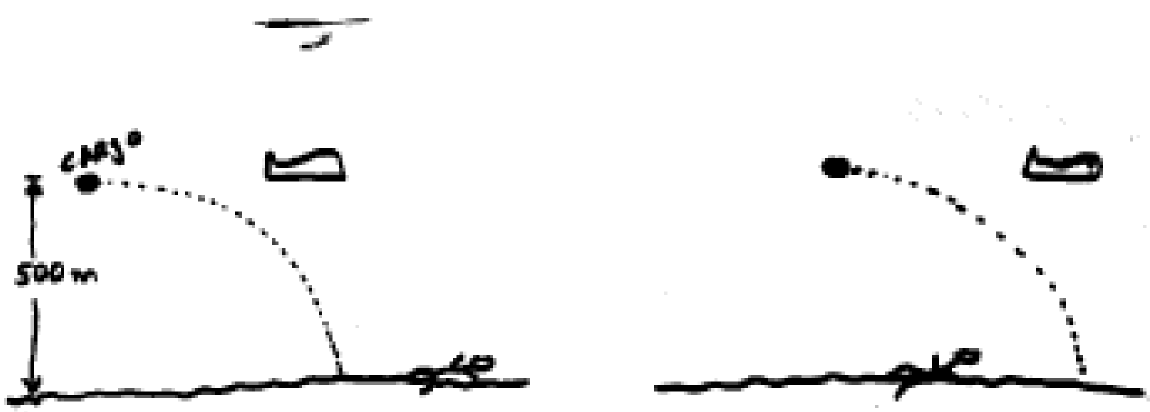
$$y = y_0 + (v_0 \sin\theta)\, t - \tfrac{1}{2} g t^2$$

$$v_Y = (v_0 \sin\theta) - g t$$

EXAMPLE:

A rescue plane flies at 55 m/s (198 Km/h) and at a constant height of h= 500 m. The objective is to deliver a rescue cargo over a victim on the land.



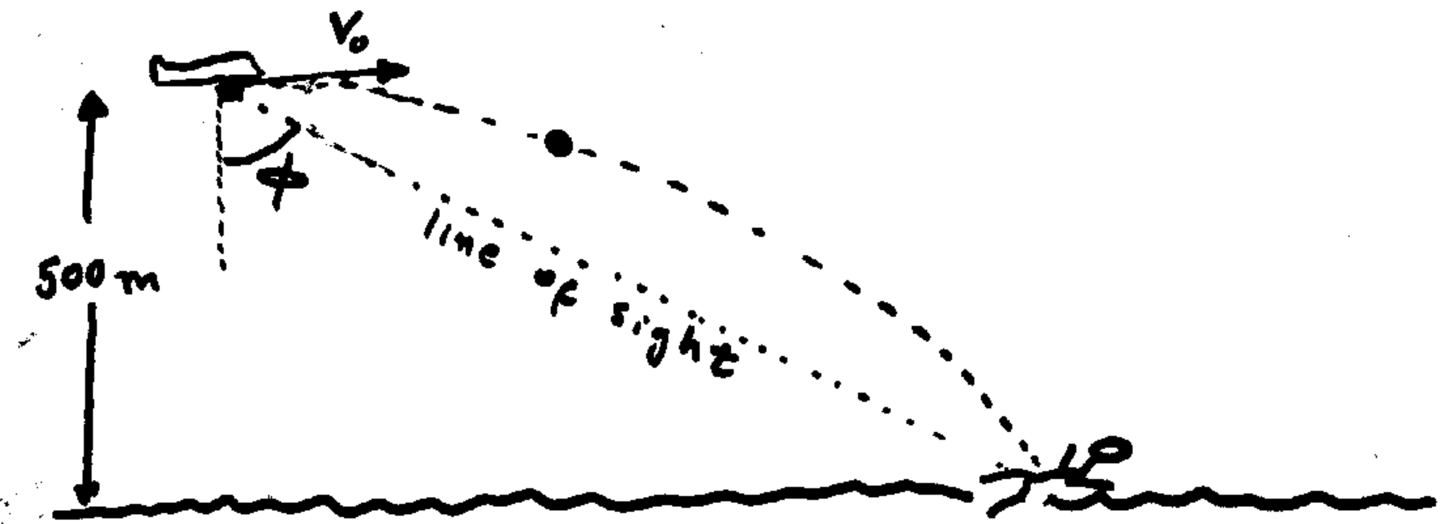


cargo released too early

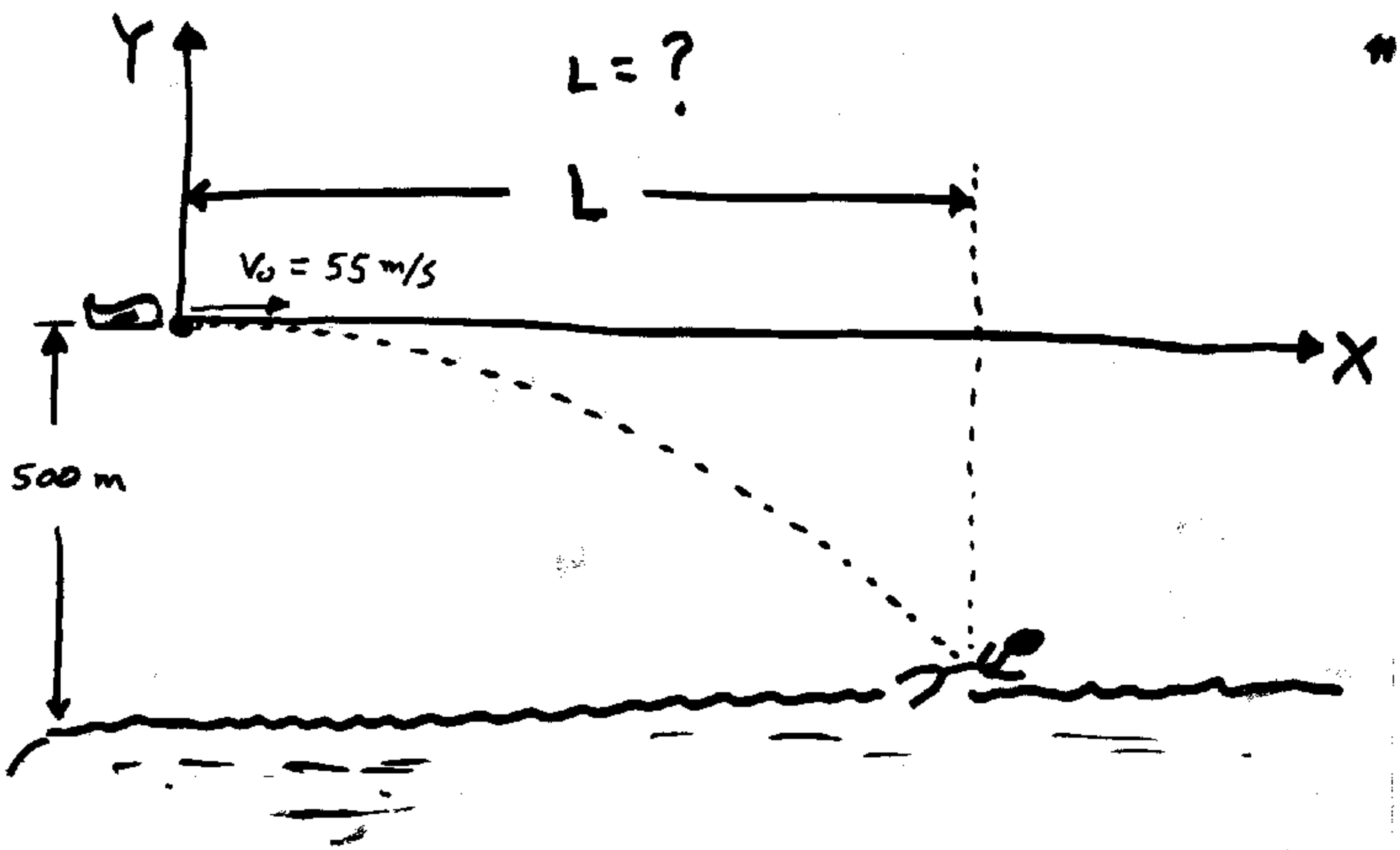
cargo released too late.

what should be the perfect timing for the cargo to be released ?

An equivalent way to post this question is as follows:



What should be angle ϕ (angle made by the pilot's line of sight when looking at the victim) at the instant that the cargo should be released ?



NOTICE: $V_{0x} = 55$ m/s $\quad V_{0y} = 0$

$x_0 = 0 \quad Y_0 = 0$

$$x = 55\tfrac{m}{s}\,t$$

$$y = -\tfrac{1}{2} g t^2$$

Notice: when the cargo reaches the victim $y = -500$ m

So, we can find how long the cargo takes to travel 500 m

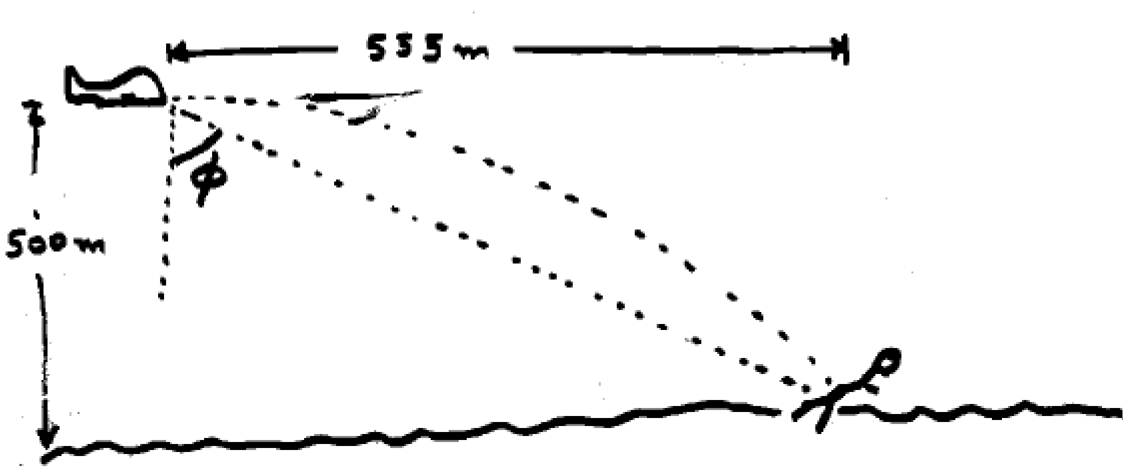
$$(-500) = -\tfrac{1}{2} g t^2 \Rightarrow t = 10.1 \text{ seconds}$$

During this time the cargo would travel a horizontal distance given by,

x = (55 m/s)x (10.1 s) = 555.

Notice, this is then the distance L at which the cargo should be released.

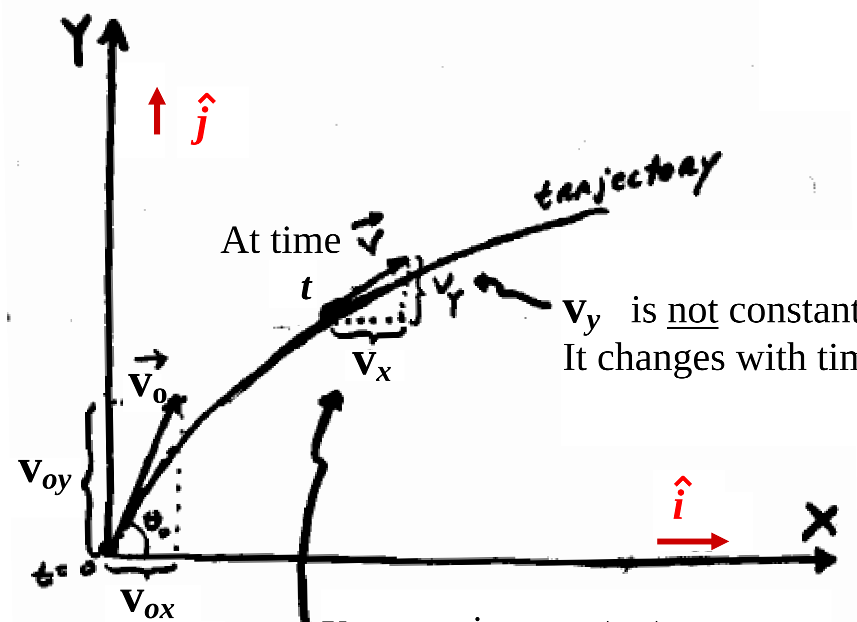
Therefore



$$\tan\phi = \frac{555}{500} \Rightarrow \phi \approx 48^{\circ}$$

Question: What is the vector velocity of the cargo just before it arrives to the floor?

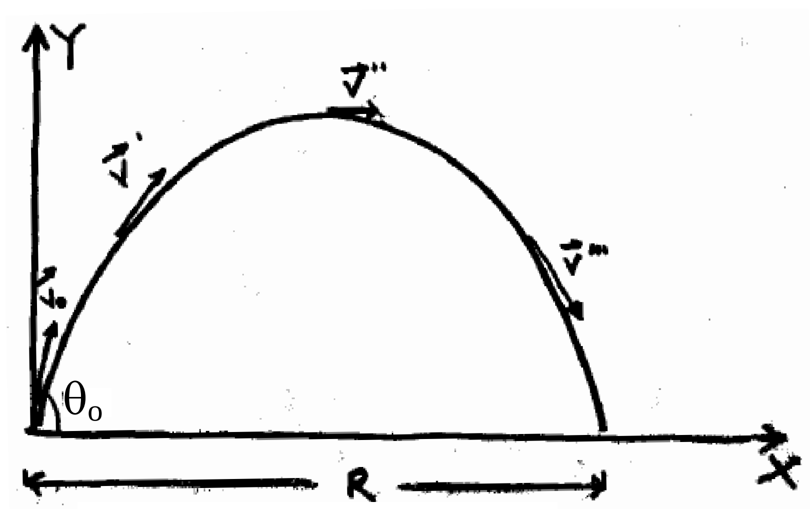
The projectile motion



$\mathbf{v}_x$ remains constant
and equal to $\mathbf{v}_{ox}$
throughout the motion

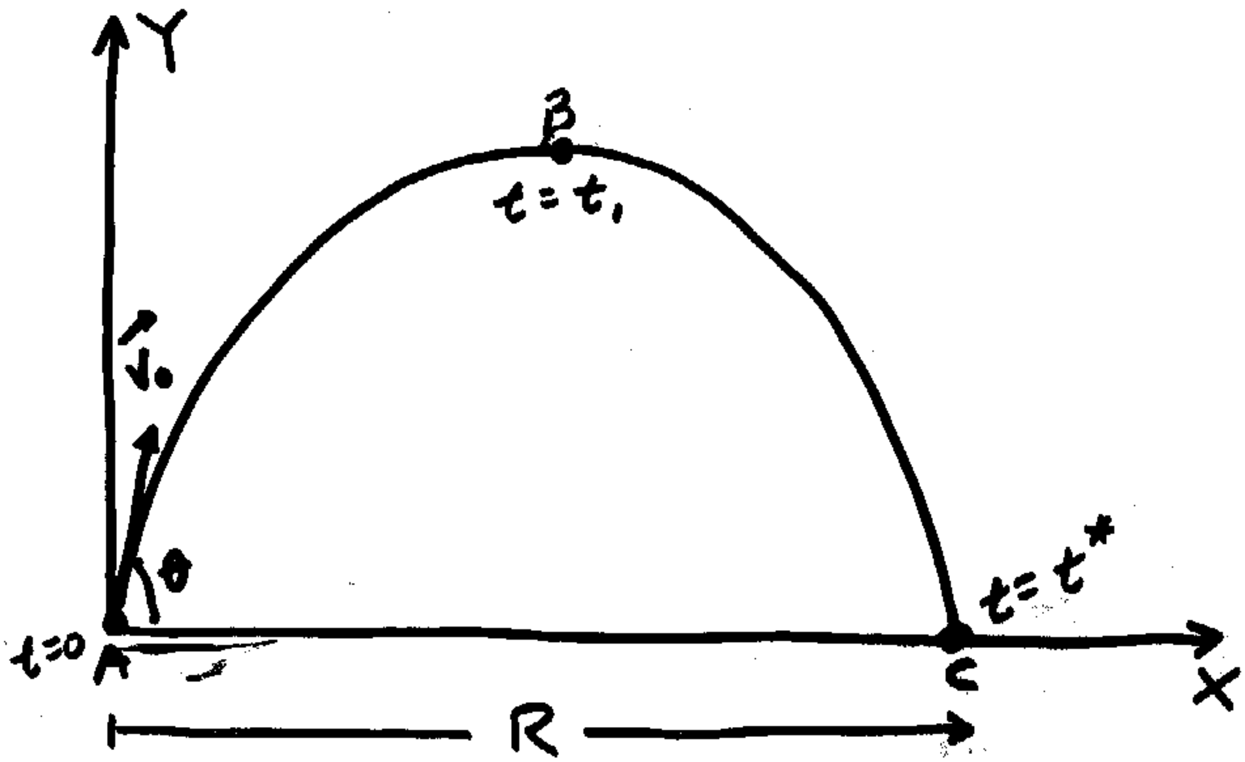
$$\vec{\mathbf{v}}_o = \mathbf{v}_{ox}\,\hat{i} + \mathbf{v}_{oy}\,\hat{j}$$

Constant vector



Horizontal range

$R = ?$ "HORIZONTAL RANGE"



Time of flight : t^*

$$t^* = \underbrace{t_{\text{bottom to top}}}_{t_1} + \underbrace{t_{\text{top to bottom}}}_{t_2}$$

$$= 2\, t_{\text{bottom to top}}$$

$$= 2\, t_1$$

At the top (point B on previous graph)

$$\left.\begin{array}{l} V_y = 0 \\ t = t_1 \end{array}\right\} \Rightarrow \quad \begin{array}{c} V_y = V_{0y} - g\,t \\ \downarrow \qquad\qquad \downarrow \\ 0 = V_{0y} - g\,t_1 \end{array}$$

$$\Rightarrow \quad t_1 = \frac{V_{0y}}{g} = \frac{V_0 \sin\theta_0}{g}$$

So $t^* = 2t_1$ ⟵ time of flight

$$= 2\frac{V_0 \sin\theta_0}{g}$$

Then

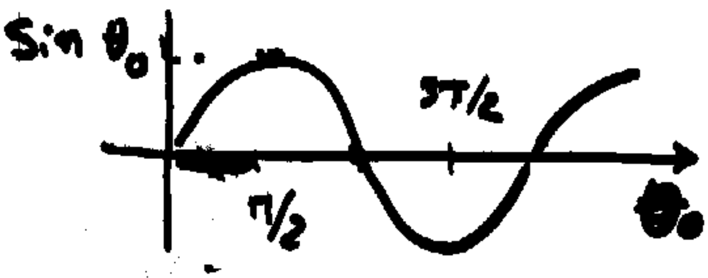
$$R = V_{0x} \cdot t^*$$

$$= (V_0 \cos\theta_0) \cdot (t^*)$$

$$= \frac{2V_0^2 \sin\theta_0 \cos\theta_0}{g}$$

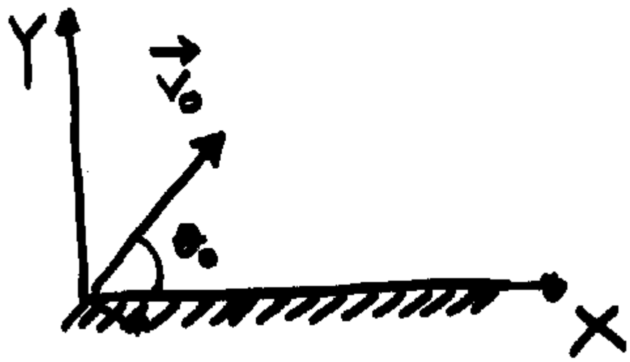
using $\sin 2\theta = 2\sin\theta\cos\theta$

$$\boxed{R = \frac{V_0^2}{g}\sin 2\theta_0}$$



But for the "projectile motion" case we are interested in the interval

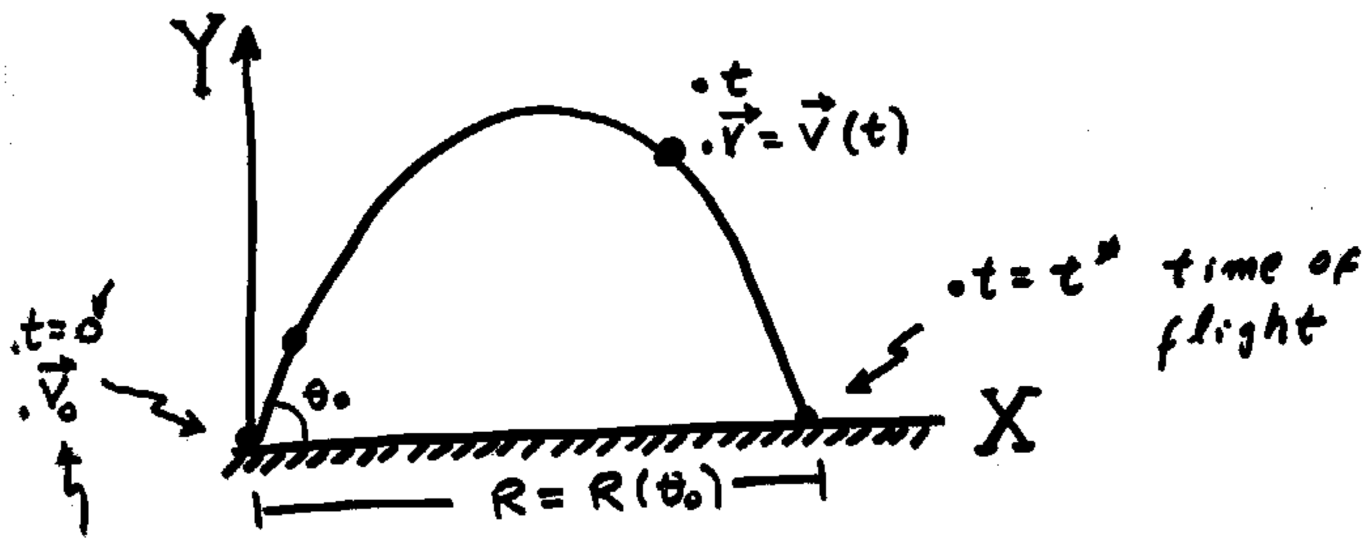
$$0 < \theta_0 \leq \frac{\pi}{2}$$



Note: $\vec{V}_0$ — initial velocity vector

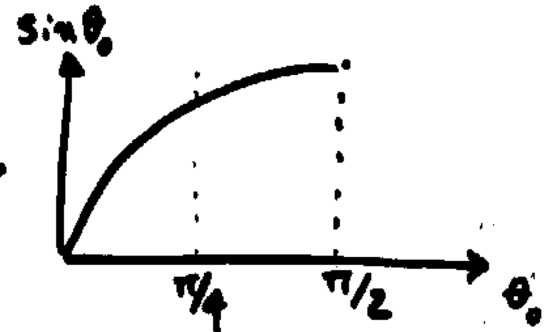
V_0 = magnitude of $\vec{V}_0$

$= |\vec{V}_0|$

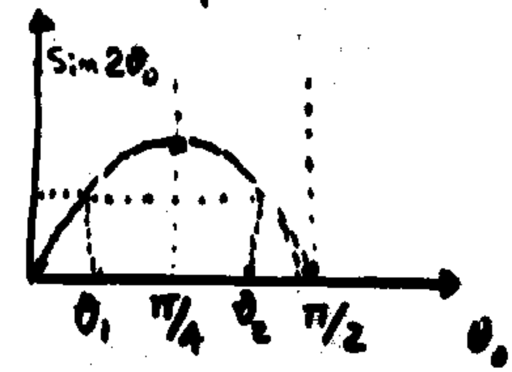


We have

$$t^* = \frac{2V_0}{g} \cdot \sin\theta_0$$

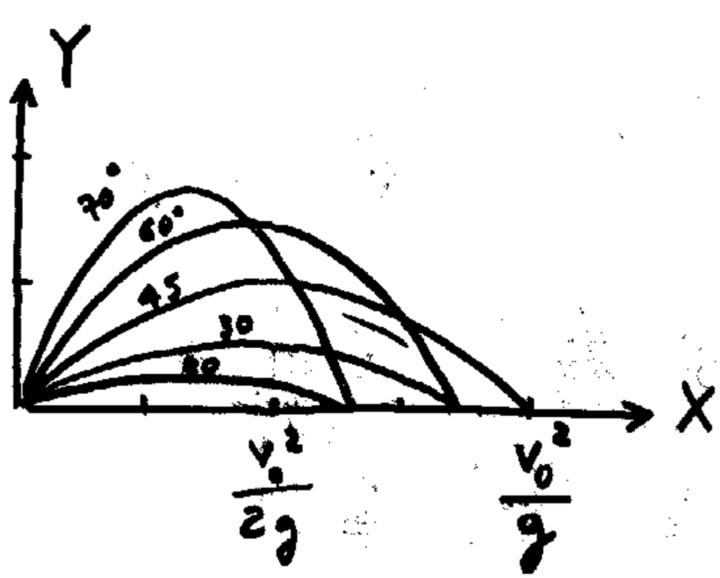


$$R = \frac{V_0^2}{g} \cdot \sin 2\theta_0$$



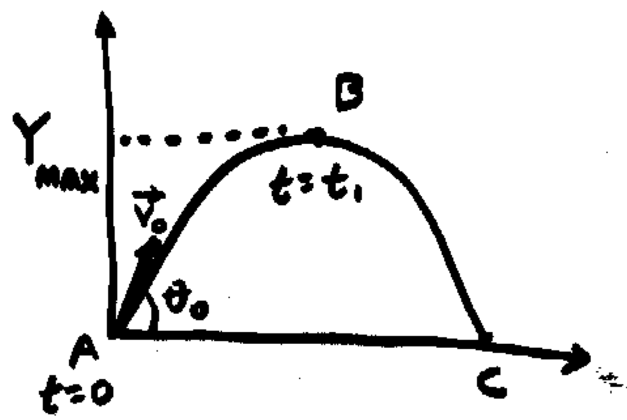
Notice in the last graph, that for 2 different angles θ_0 we obtain the same range

Question: For which angle θ_0 do we obtain the maximum range?



What is the maximum height reached by the projectil ?

$$y = V_{0y} t - \frac{1}{2} g t^2$$



At point B $t_1 = ?$

$V_y = 0$

$$t = t_1 = \frac{V_{0Y}}{g}$$

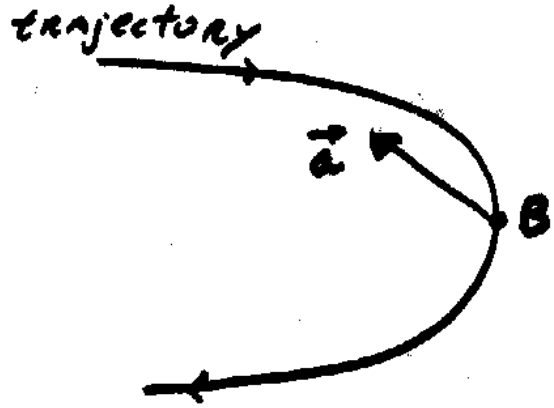
$$y \equiv \underline{y_{MAX}}$$

$$y_{max} = V_{0Y} t_1 - \frac{1}{2} g t_1^2$$

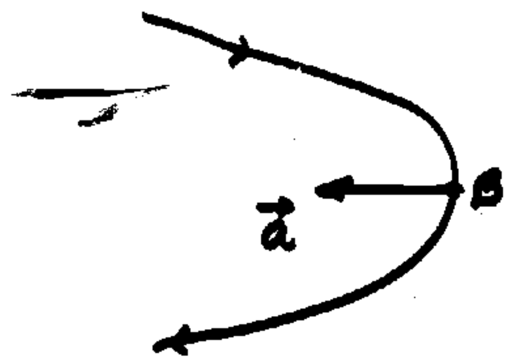
$$= V_{0Y} \frac{V_{0Y}}{g} - \frac{1}{2} g \left(\frac{V_{0Y}}{g}\right)^2 = \frac{1}{2} \frac{V_{0Y}^2}{g}$$

$$\boxed{Y_{MAX} = \frac{1}{2} \frac{(V_0 \sin\theta_0)^2}{g} \qquad Y_{MAX} = \frac{V_0^2}{2g} \sin^2\theta_0}$$

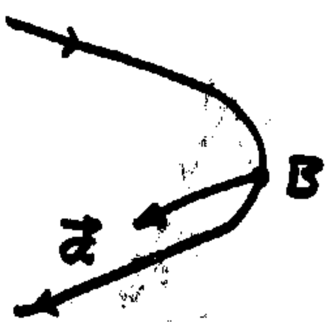
EXERCISE



(1)



(2)



(3)

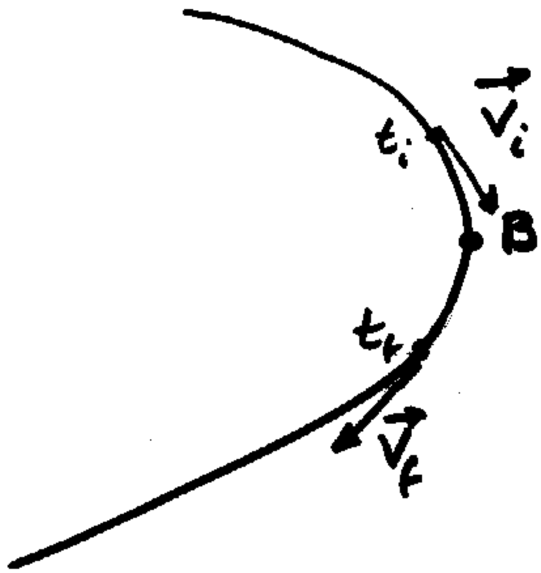
Which one is the correct vector acceleration when the particle passes by the position B ?

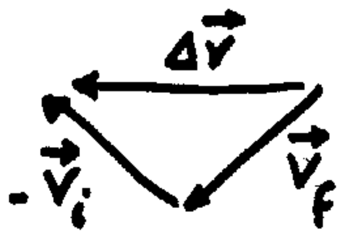
we need

$$\Delta\vec{v} = \vec{v}_f - \vec{v}_i$$

$$= \vec{v}_f + (-\vec{v}_i)$$

$$\vec{a} = \frac{\Delta\vec{v}}{\Delta t}$$

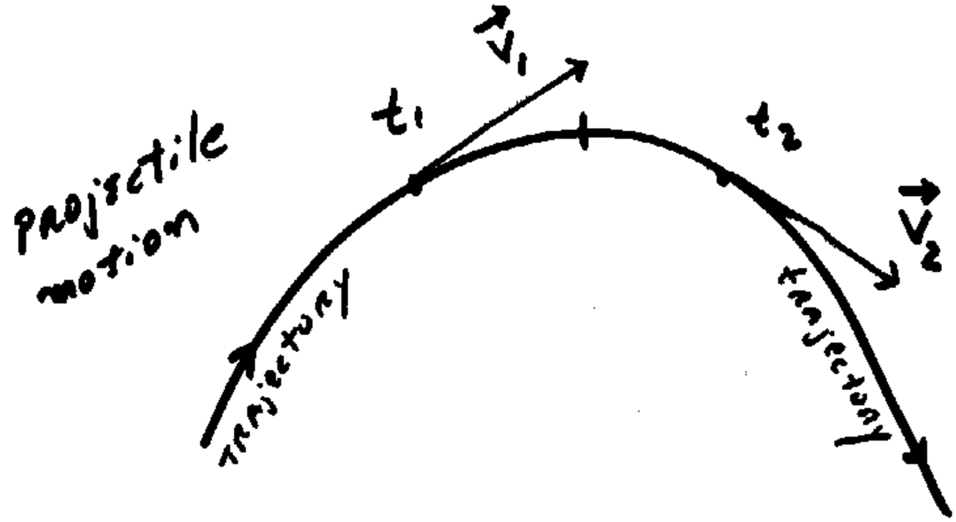




$\hookrightarrow |\vec{v}_i| < |\vec{v}_f|$

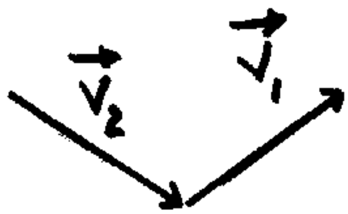
$|\vec{v}_i| = |\vec{v}_f|$

$|\vec{v}_i| > |\vec{v}_f|$



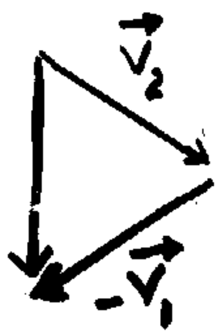
$\Delta t = t_2 - t_1$

$\Delta \vec{v} = \vec{v}_2 - \vec{v}_1$



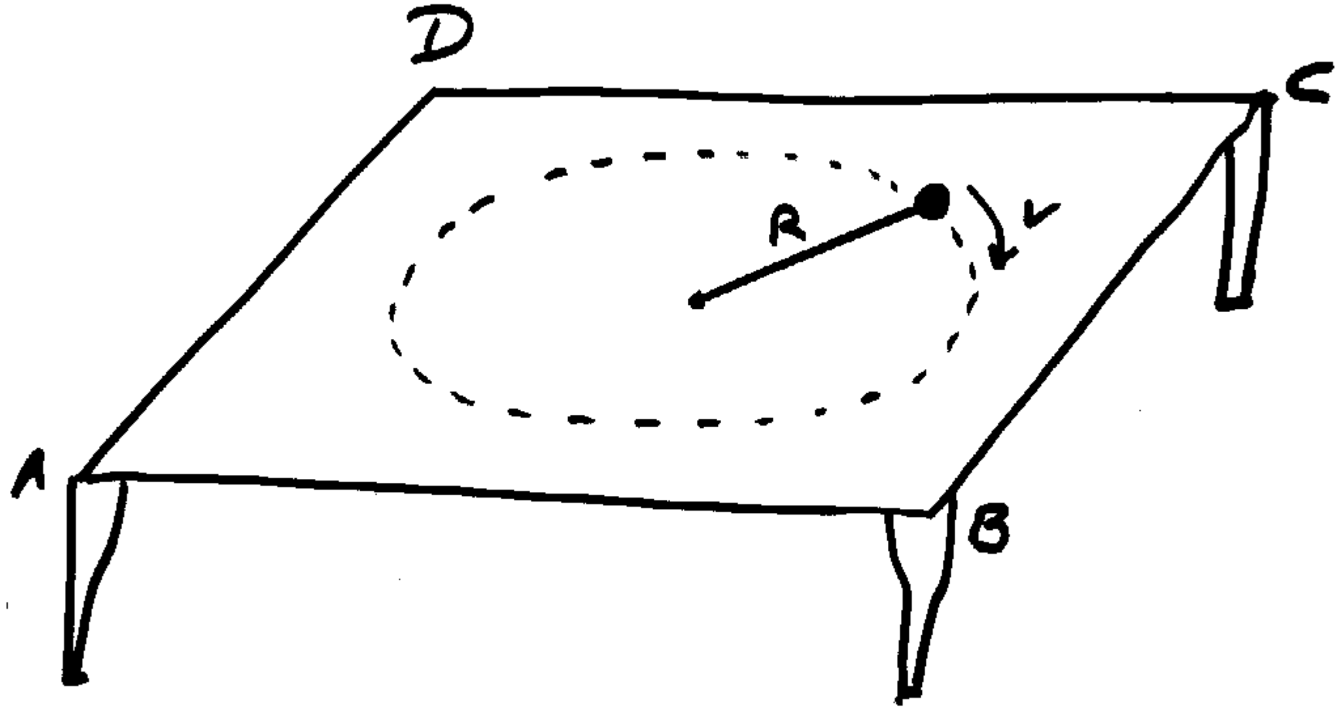
$\Delta \vec{v} =$

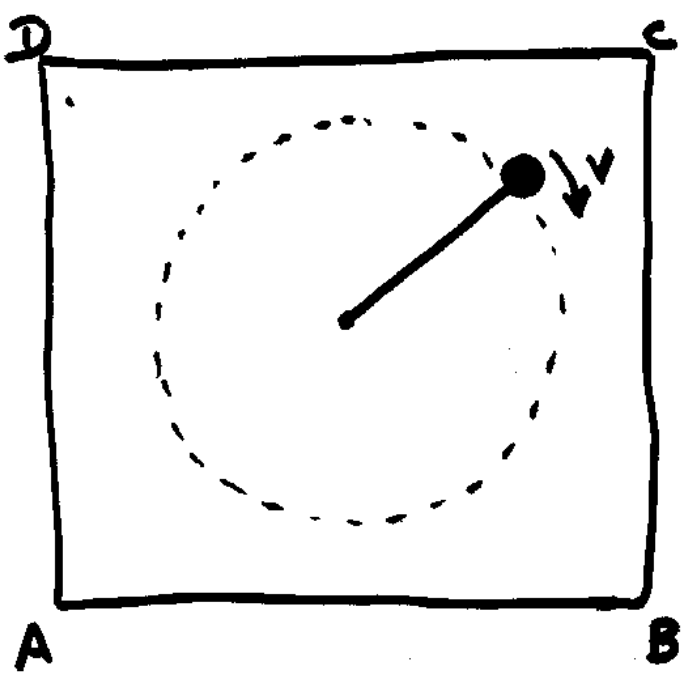
$= \vec{v}_2 - \vec{v}_1$



QUESTION: what is the AVERAGE ACCELERATION of the particle between t_1 and t_2?

CIRCULAR MOTION





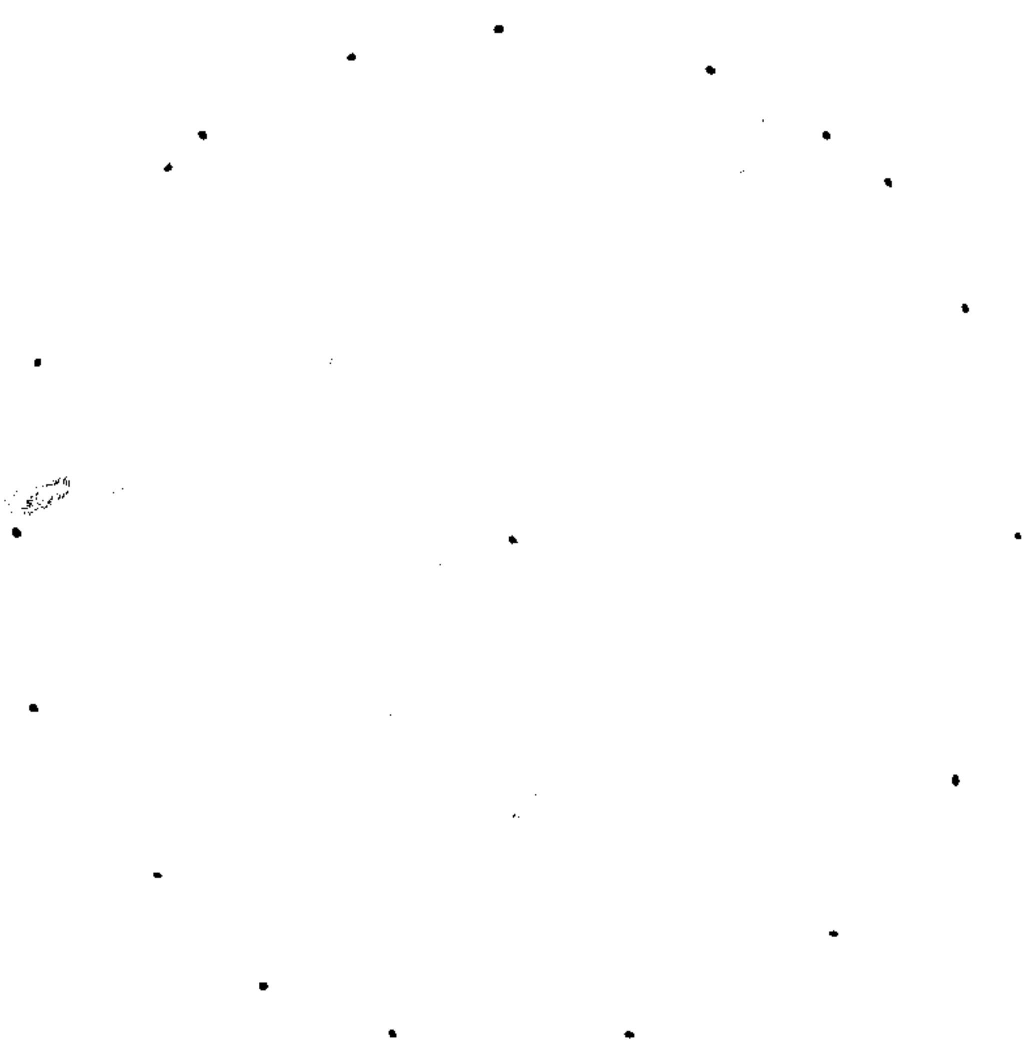
TOP VIEW

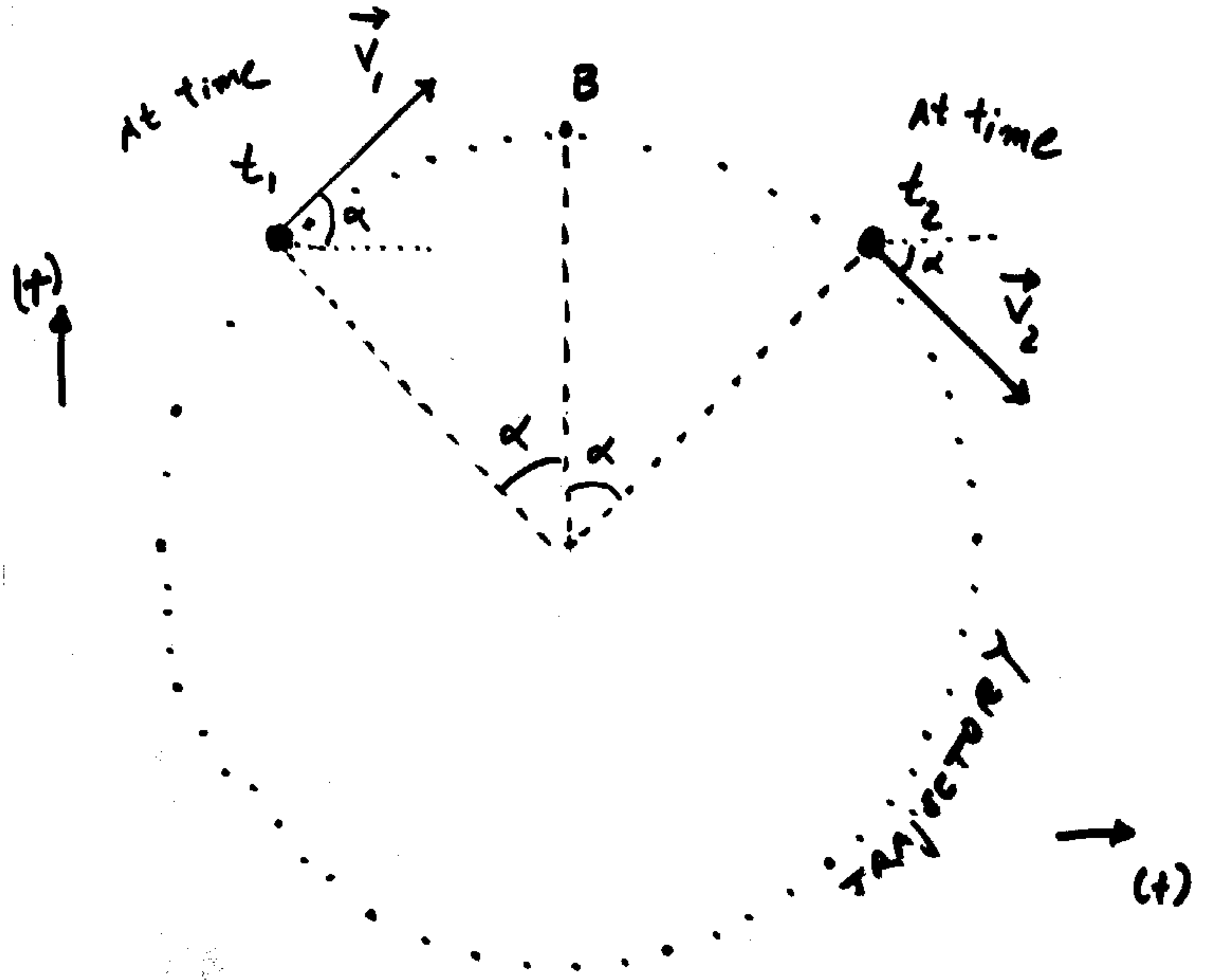
Uniform Circular Motion

Trajectory — circumference

Speed $|\vec{v}|$ — constant $\equiv v$

velocity $\vec{v}$ — <u>not</u> constant

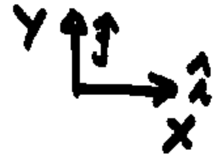




α : A VERY small angle

$|\vec{V}_1| = |\vec{V}_2| \equiv v$ ← magnitude of the velocity

$$\vec{V}_2 = v\cos\alpha \cdot \hat{\imath} + v\sin\alpha \cdot (-\hat{\jmath})$$



$$= v\cos\alpha\, \hat{\imath} - v\sin\alpha \cdot \hat{\jmath} \qquad \text{Eq } ①$$

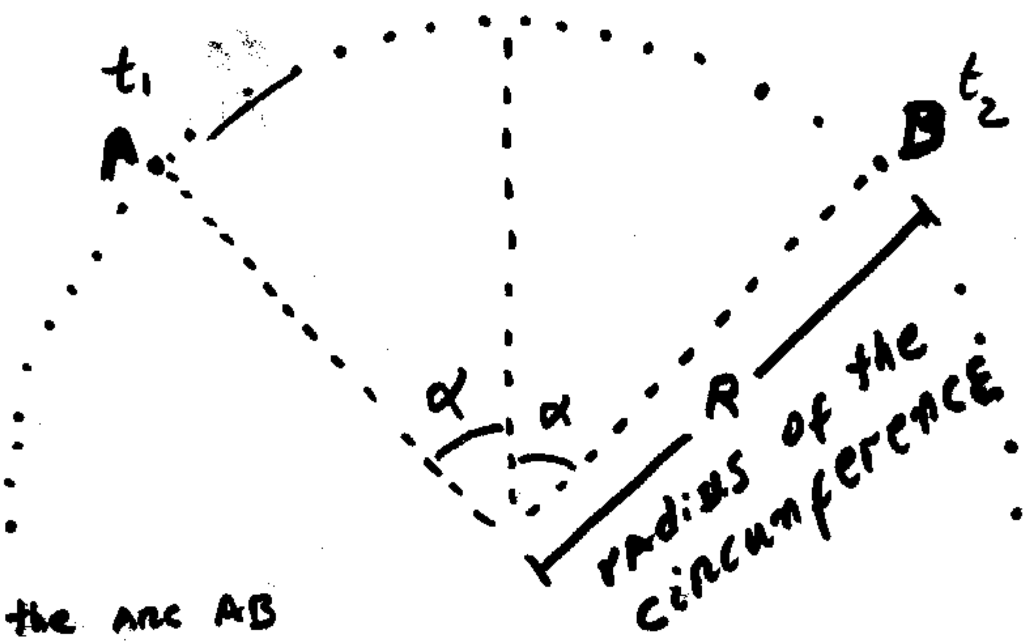
$$\vec{V}_1 = v\cos\alpha\, \hat{\imath} + v\sin\alpha \cdot \hat{\jmath} \qquad \text{Eq } ②$$

From ① and ② we obtain

$$\vec{V}_2 - \vec{V}_1 = -2v\sin\alpha\; \hat{\jmath} = \Delta\vec{v} \qquad \text{Eq } ③$$

NEXT TASK: TO EXPRESS $\Delta t = t_2 - t_1$ in terms of α and v

By definition:



$$\underbrace{2\alpha}_{\text{angle in radiants}} = \frac{\text{arc } AB}{R}$$

$$2\alpha = \frac{\text{length of the arc } AB}{R}$$

But

length of ARC AB $= v \cdot \Delta t$ $\qquad \Delta t = t_2 - t_1$

So

$$2\alpha = \frac{v \cdot \Delta t}{R}$$

From which we obtain

$$\Delta t = \frac{2\alpha \cdot R}{v} \qquad \text{Eq } 4$$

FROM 3 and 4

$$\frac{\Delta \vec{v}}{\Delta t} = \frac{-2v \sin\alpha \,\hat{j}}{2\alpha \cdot R/v}$$

$$= -\frac{v^2}{R} \underbrace{\frac{\sin\alpha}{\alpha}}_{1} \hat{j}$$

AVERAGE ACCELERATION between t_1 and t_2

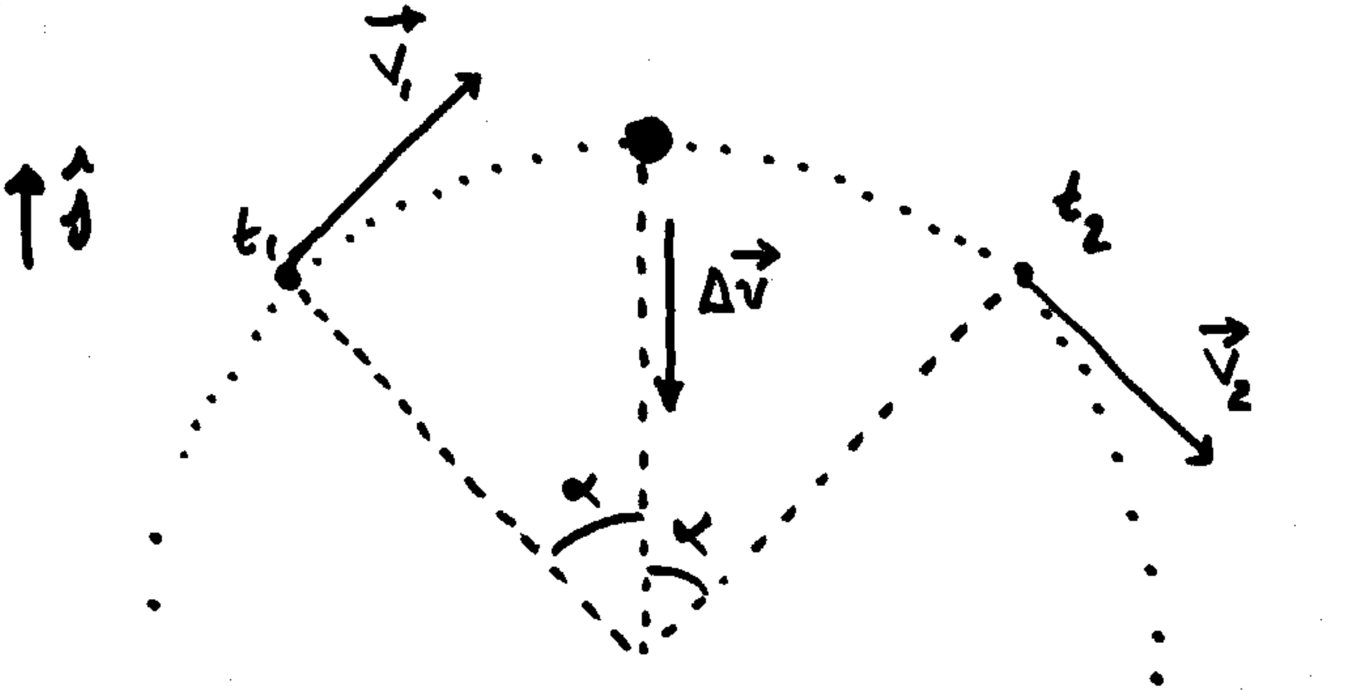
fill the following table

α	$\sin\alpha$	$\frac{\sin\alpha}{\alpha}$
1 rad		0.841
0.5 rad		0.958
0.25		0.989
0.1		0.998
0.05		0.9996
0.025		0.9998
0.01		
0.005		
$\downarrow$		$\downarrow$
0		1

what we are doing is to find

$$\lim_{\alpha \to 0} \frac{\sin\alpha}{\alpha} = 1$$

$$\Delta t = t_2 - t_1$$



Notice

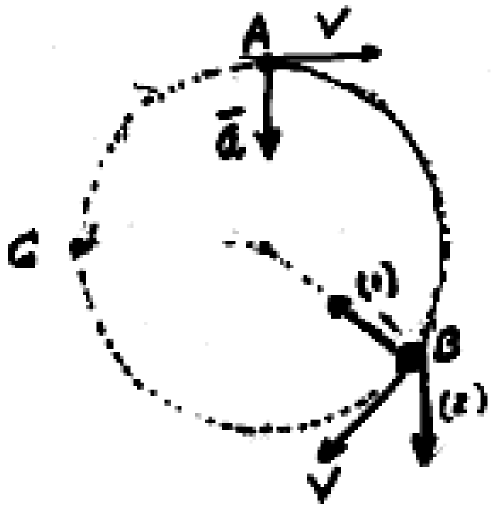
$$\lim_{\alpha \to 0} \underset{\text{equivalent}}{\Longleftrightarrow} \Delta t \to 0$$

$$\underbrace{\lim_{\alpha \to 0} \frac{\Delta \vec{V}}{\Delta t}}_{\vec{a}} = \lim_{\alpha \to 0} \frac{-v^2}{R} \underbrace{\frac{\sin\alpha}{\alpha}}_{1} \hat{j}$$

$$\boxed{\vec{a} = -\frac{v^2}{R}\hat{j}} \qquad \vec{a} = \frac{v^2}{R}(-\hat{j})$$

instant acceleration

Had we chosen a different point along the circumference, we would have obtained:



Acceleration in B ?

(1) or (2) ?