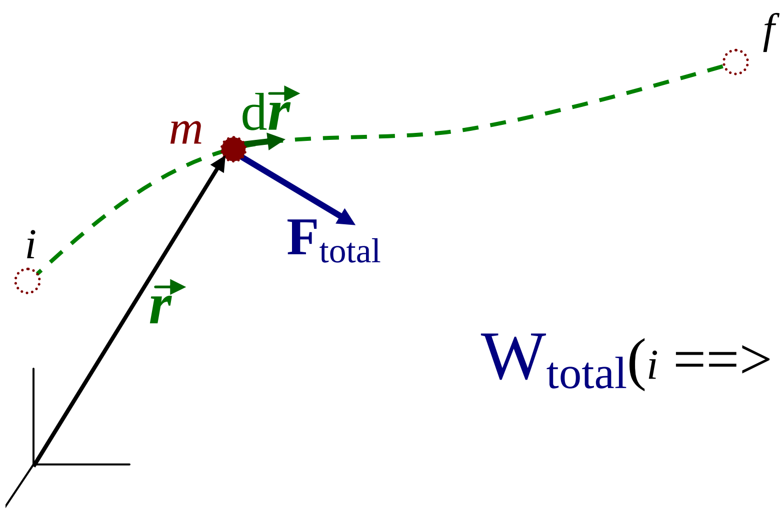
Work – Kinetic energy theorem
Case: 3D motion

Consider a particle of mass *m*, which moves from the initial position indicated by "*i*" to the final position "j" along the dashed curve shown in the figure. During its motion, a net force $\vec{\mathbf{F}}_{\text{total}}$ acts upon the particle.



$$W_{\text{total}}(i \Longrightarrow f) = \int_i^f \vec{\mathbf{F}}_{\text{total}} \cdot d\vec{\boldsymbol{r}}$$

$\vec{\boldsymbol{r}}$ is the vector position of the particle

$$\int_i^f \vec{\mathbf{F}}_{\text{total}} \cdot d\vec{\boldsymbol{r}} = \int_i^f m\,\vec{\mathbf{a}} \cdot d\vec{\boldsymbol{r}}$$

But notice,

$$\vec{\mathbf{a}} \cdot d\vec{\boldsymbol{r}} = \frac{d\,\vec{\mathbf{v}}}{dt} \cdot d\vec{\boldsymbol{r}} = d\vec{\mathbf{v}} \cdot \frac{d\,\vec{\boldsymbol{r}}}{dt} = d\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}$$

$$= \vec{\mathbf{v}} \cdot d\vec{\mathbf{v}} = \frac{1}{2} d(\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}) = \frac{1}{2} d(v^2)$$

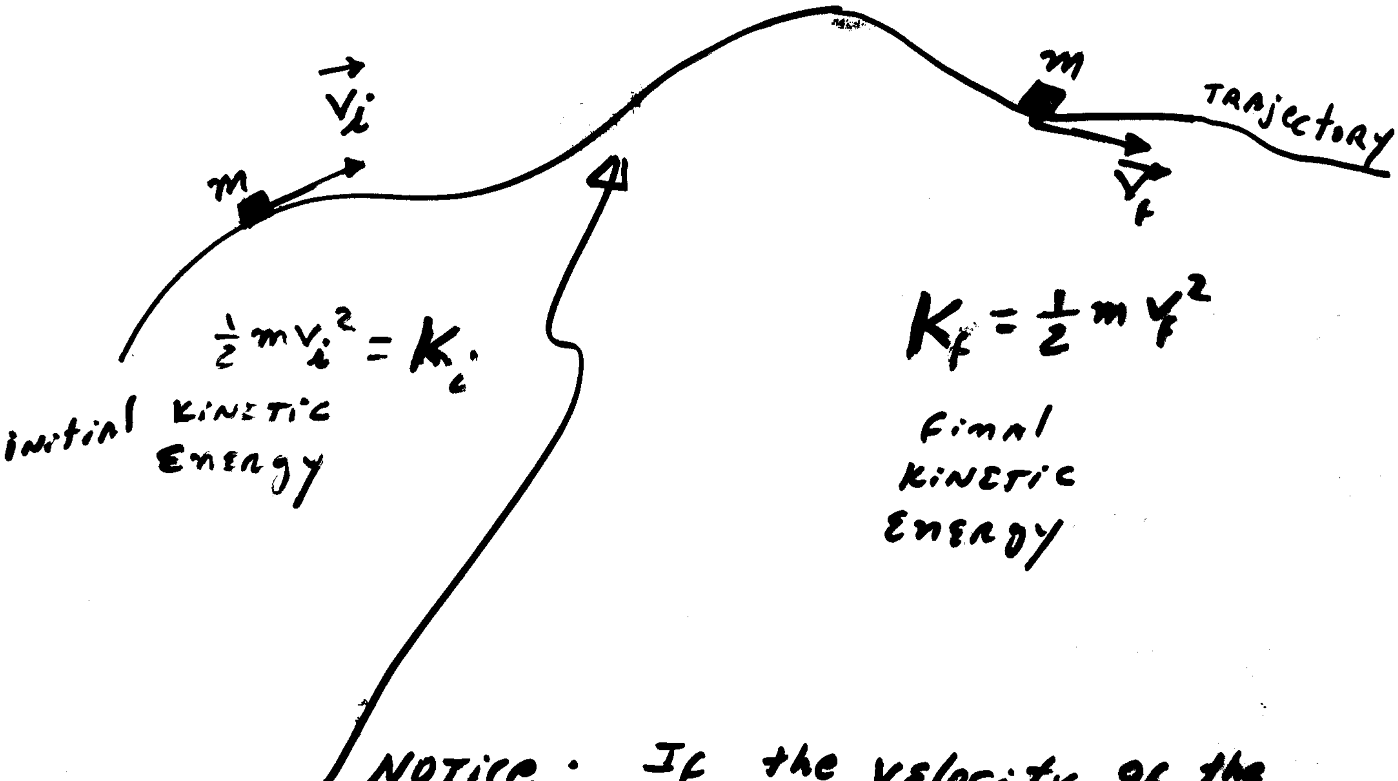
$$\int_i^f \vec{\mathbf{F}}_{\text{total}} \cdot d\vec{\boldsymbol{r}} = \frac{1}{2}\, m \int_i^f d(\mathrm{v}^2)$$

$$= \frac{1}{2}\, m \left[(\mathrm{v}_f)^2 - (\mathrm{v}_i)^2 \right] = \mathrm{K}_f - \mathrm{K}_i$$

That is,

$$\boxed{\mathrm{W}(i \Longrightarrow f) = \mathrm{K}_f - \mathrm{K}_i}$$

MOTION IN 3D



NOTICE: If the velocity of the block m is changing, is <u>because</u> some forces are acting on m

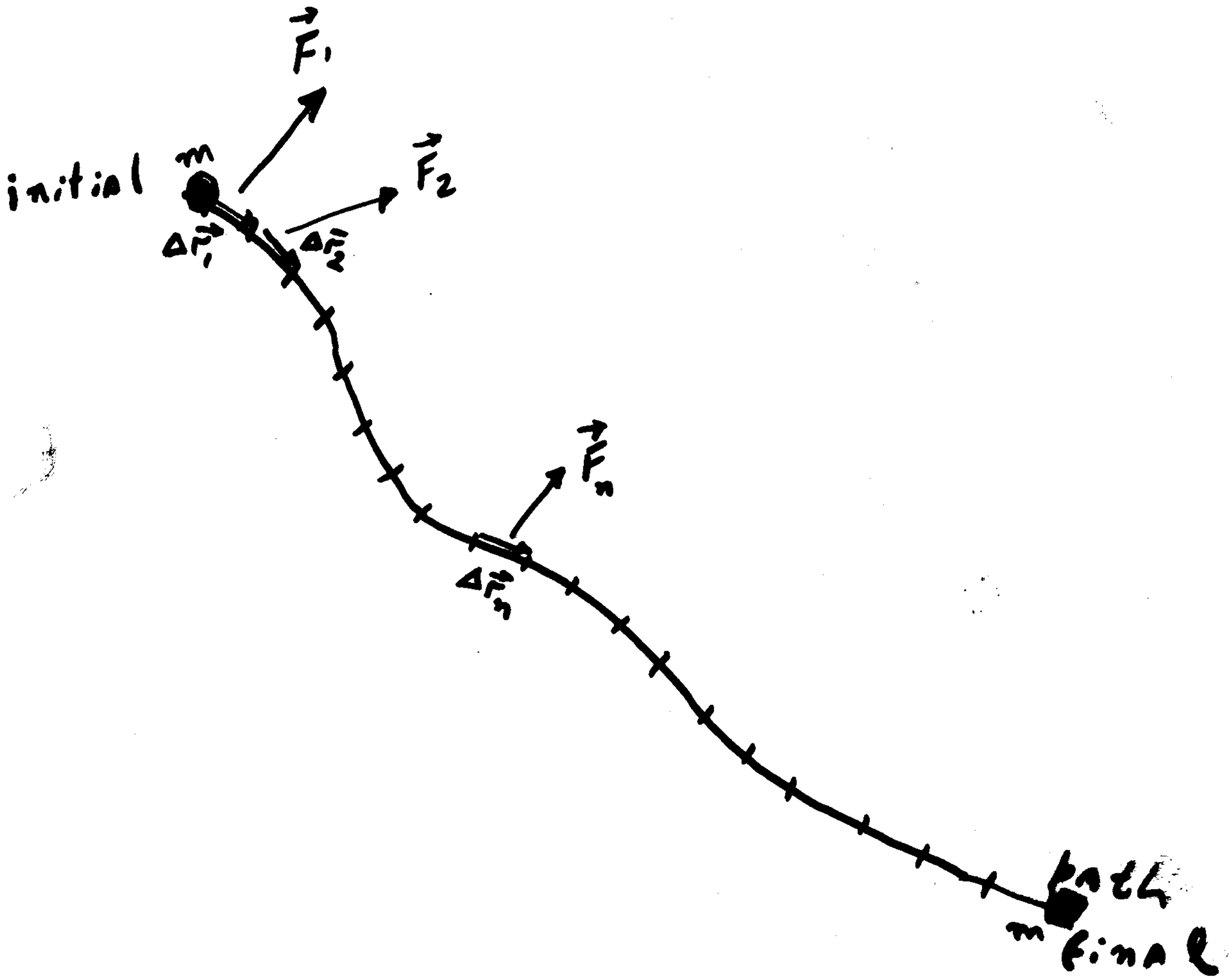
$$\Delta K = K_f - K_i$$

$$= \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = W$$

W: work done by <u>all</u> the forces acting on m

How to evaluate W in a 3-D motion

a) Break the path down into small segments.



b) Evaluate the work ~~at~~ each segment

$$\Delta W_n = \vec{F}_n \cdot \Delta\vec{r}_n$$

c)

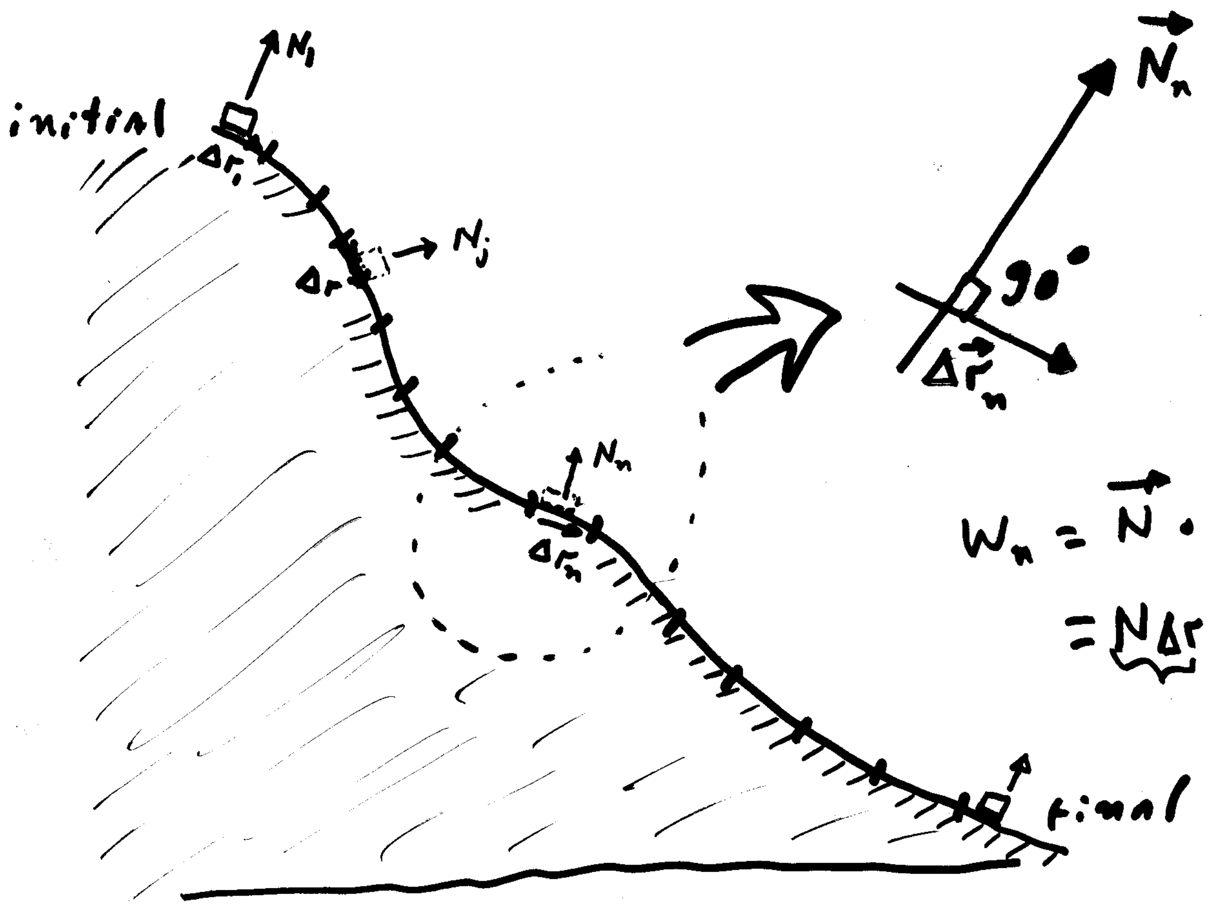
Add up all the ΔW_n

$$W = \Delta W_1 + \Delta W_2 + \ldots + \Delta W_n + \ldots$$

$$= \vec{F}_1 \cdot \Delta \vec{r}_1 + \vec{F}_2 \cdot \Delta \vec{r}_2 + \ldots + \vec{F}_n \cdot \Delta \vec{r}_n + \ldots$$

$$\boxed{W = \sum_{\text{initial}}^{\text{final}} \vec{F}_n \cdot \Delta \vec{r}_n}$$

Example: Calculate the work done by the Normal force



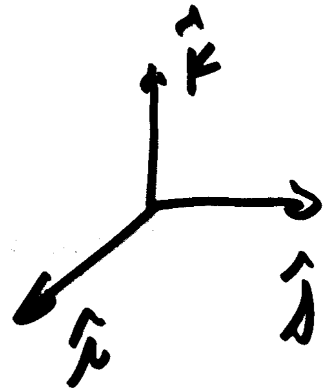
$$W_n = \vec{N} \cdot \Delta \vec{r}_n$$

$$= \underbrace{N \Delta r}\ \underbrace{\cos 90°}_{=0}$$

$$\underset{\text{work done by the normal force}}{W} = 0$$

$$m\vec{g} = -mg\,\hat{k}$$

$$\Delta\vec{r} = \Delta x\,\hat{i} + \Delta y\,\hat{j} + \Delta z\,\hat{k}$$



$$m\vec{g}\cdot\Delta\vec{r} =$$

$$= \underbrace{-mg}\,\hat{k}\cdot\left(\Delta x\,\hat{i} + \Delta y\,\hat{j} + \Delta z\,\hat{k}\right)$$

$$= -mg\,\Delta x\,\cancel{\hat{k}\cdot\hat{i}} +$$

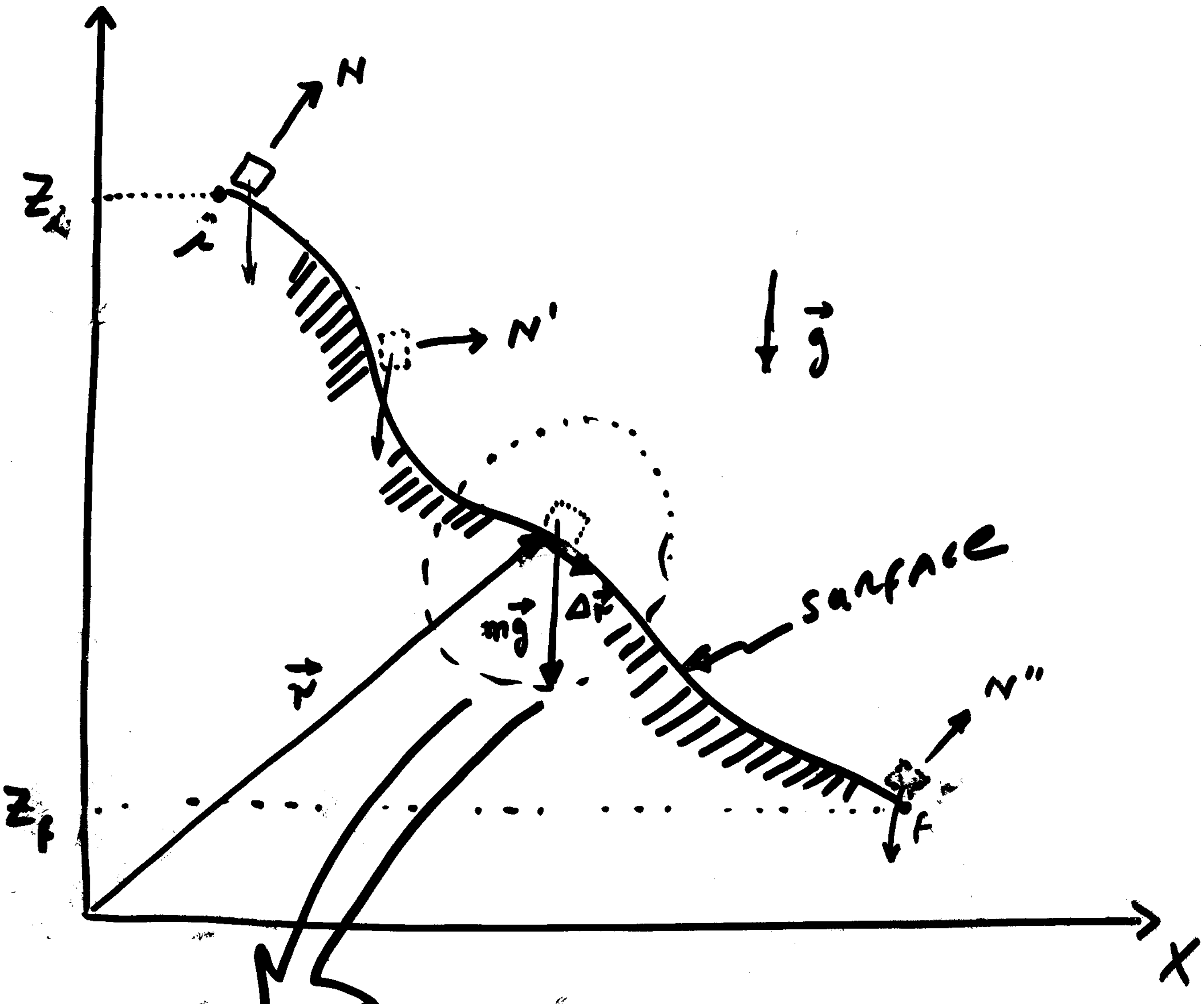
$$-mg\,\Delta y\,\cancel{\hat{k}\cdot\hat{j}} +$$

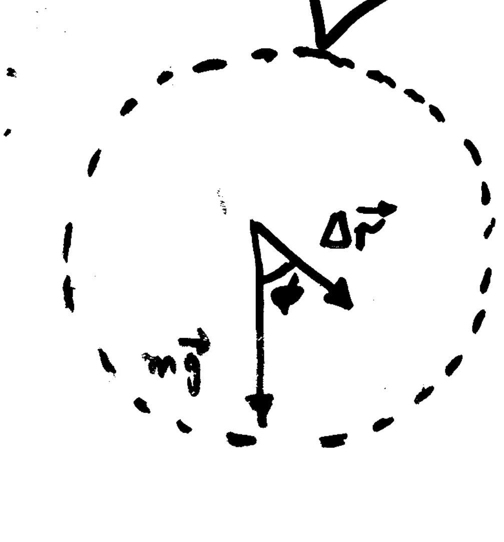
$$-mg\,\Delta z\,\underbrace{\hat{k}\cdot\hat{k}}_{=1}$$

$$= -mg\,\Delta z$$

$$\vec{A}\cdot\vec{B} = A\,B\,\underbrace{\cos\theta}$$

Example Calculate the work done by the gravitational force

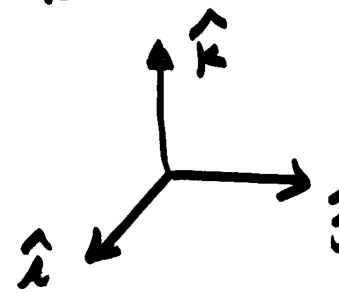




$$\Delta W = \vec{F} \cdot \Delta\vec{r}$$
$$= m\vec{g} \cdot \Delta\vec{r}$$

$$\Delta\vec{r} = \Delta x\,\hat{i} + \Delta y\,\hat{j} + \Delta z\,\hat{k}$$

$$m\vec{g} = -mg\,\hat{k}$$



$$\Delta W = \Delta\vec{r} \cdot m\vec{g} = -mg\,\Delta z$$

$$W_{i\to f} = \sum_i^f \Delta W = \sum_i^f -mg\,dz$$

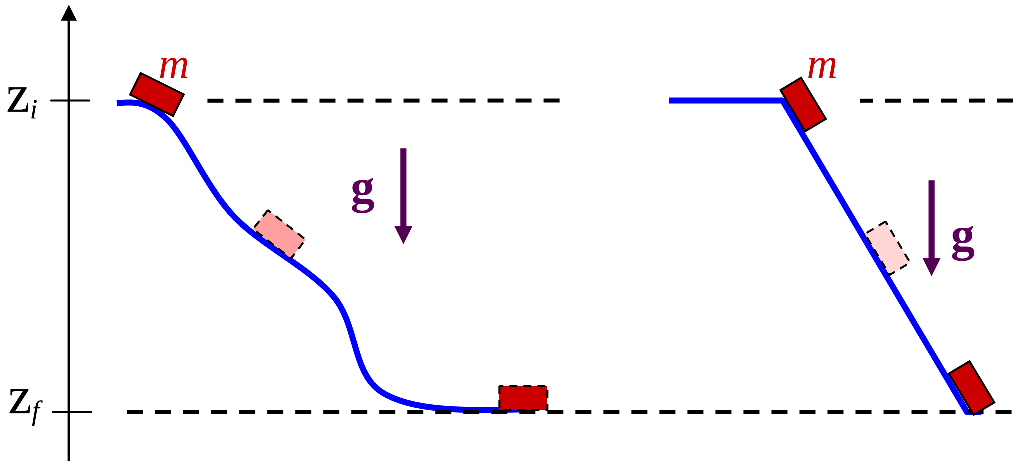
$$\boxed{W_{i\to f} = -mg(z_f - z_i)}$$

work done by the gravitational force $m\vec{g}$

/ work does not depend on the particular trajectory.

Work just depend on the z-coordinate of the initial and final position. SO: GRAVITATIONAL FORCE is CONSERVATIVE!

Notice,



In both cases, the work done by the gravitational force on the block of mass *m* is

$$W_{i \to f} = -mg\,(z_f - z_i)$$

Independent of the particular trajectory followed. It depends only on the initial and final heights.

This type of results brings us closer to the concept of mechanical energy, which will be described in the next chapter.