A. La Rosa Lecture Notes

INTRODUCTION TO QUANTUM MECHANICS

PART-II MAKING PREDICTIONS in QUANTUM MECHANICS and the HEISENBERG's PRINCIPLE

CHAPTER-5
QUANTUM BEHAVIOR of PARTICLES and the HEISENBERG'S UNCERTAINTY PRINCIPLE

5.1 Quantum behavior of particles passing through two slits

- **5.1.A The concept of Probability. An experiment with bullets**
- **5.1.B An experiment with light**
- **5.1.C An experiment with electrons**
- **5.1.C Attempts to track electrons' trajectories**

5.2 The Heisenberg's Uncertainty Principle

- **5.2.A Uncertainty in the position Δx and linear momentum Δp.**
  Example: Particles passing through a single slit.
- **5.2.B Relationship between the uncertainty of the energy content ΔE of a pulse and the time Δt required for the measurement**
  - **5.2.B.a Gratings**
    Phasors (addition of multiple waves),
    Conditions for constructive interference
    Evaluation of the sharpness of the peak of order *m*.
  - **5.2.B.b Gratings and Spectral Resolution.**
  - **5.2.B.c Condition for the minimum length time Δt required for the measurement of the energy E with a resolution ΔE.**
- **5.2.C. Uncertainty principle and the resolving power of a microscope.**
  - **5.2.C.a Image formation and the resolving power of a lens.**
  - **5.2.C.b Watching electrons through the microscope**

5.3 Interpretation of the Wavefunction

- **5.3.A The need of a wavepacket (rather than a single harmonic function) to describe a free particle**
- **5.3.B Einstein's interpretation of the granularity of the electromagnetic field**
- **5.3.C Max Born's Probabilistic Interpretation of the wavefunction**
  The concept of ensemble

5.3.D Revisiting the Heisenberg's uncertainty principle
5.3.E The Philosophy of Quantum Theory

References

Richard Feynman, "The Feynman Lectures on Physics," **Volume III**, Chapter 1.

Richard Feynman, "The Feynman Lectures on Physics," **Volume I**, Chapters 30 and 38.

CHAPTER-5
QUANTUM BEHAVIOR: The WAVE-FUNCTION and the HEISENBERG'S UNCERTAINTY PRINCIPLE

5.1 Quantum behavior of particles passing through two slits

At the atomic scale, physical phenomena behave in such a way that escapes our earlier-formed intuition (the latter developed observing macro-scale phenomena.) They are so unlike ordinary experience that give the appearance of peculiar and mysterious to everyone, both to the novice and to the experienced physicist. In consequence,

> *we have to learn quantum behavior phenomena in a sort of abstract or imaginative fashion and not by connection with our direct experience with macro-scale objects*.[1]

In this section we will describe one of those phenomena, namely, the behavior of electrons passing through a couple of slits, which turns out to be absolutely impossible to explain in classical terms, and which has in it **the heart of quantum mechanics**.
In general, a striking new feature in quantum mechanics problem refers to the impossibility in obtaining a perfect knowledge of all the physical variables that describe the motion of an atomic object (electrons, in the case of the experiment to be described below). There is always an inherent uncertainty in the variables.

At the beginning, we will get the impression that the lack of certainty in deterministically describing the motion of electrons has the appearance of a mystery. This mysterious phenomenon may constitute the only 'mystery' in quantum mechanics. Thus, it is worth to consider having an early exposure to it. But, paraphrasing Feynman,

> *we will not be able to explain how it works,*
> *we will tell you how it works*.[2]

That is to say, we will provide a specific mathematical procedure that when applied to a given problem (an electron passing through a couple of slits, for example) it predicts an outcome that happens to coincide with the experimental results.

In the following sections, we will contrast the quantum behavior of atomic particles (electrons) with the behavior of macroscopic particles (gun bullets), and make this difference in the context of the more familiar behavior of waves (using light interference as a specific example.)

5.1.A The concept of Probability. An experiment with bullets

A gun shoots a stream of bullets, randomly and over a large angular spread. Between the gun and the screen there exists a wall with two slits big enough to let one bullet to pass through. A detector at the screen serves to count the number of bullets arriving at each position x.

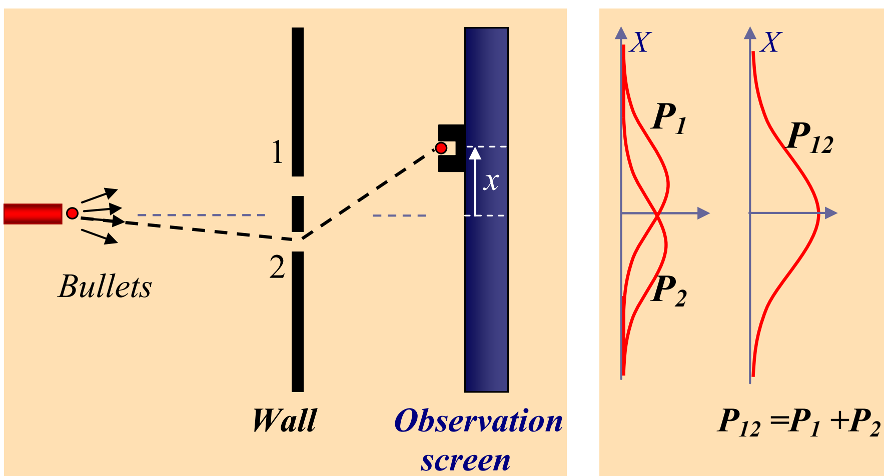


Fig. 5.1 Schematic of the experimental setup to monitor the passing of bullets through two slits. The traces on the right show the distribution of bullets arriving at the screen corresponding to different test scenarios: one of the apertures blocked (P_1 and P_2), or both unblocked (P_{12}.)

Some bullets may go directly straight through any of the holes, or bounce on the edges, etc. Thus there is some uncertainty about the exact position "x" at which a particular bullet will arrive at the screen.

One way to deal with uncertain quantities is to use the concept of probability. Let's introduce the concept of probability through the analysis of different possible scenarios that can occur when using the experimental setup presented above.

- Let's block hole #1. What would be the probability $P_2(x)$ that a bullet passing through the hole 2 arrives at the position x?

  The answer requires to make an experimental procedure:

  If out of a total number N_{total} of incident bullets, the number of bullets hitting the screen at x is $N_2(x)$, then the probability $P_2(x)$ is defined as the ratio $P_2(x) \equiv N_2(x)/N_{total}$. But for the concept of probability to acquire a useful meaning the total trial number N_{total} has to be very large. Symbolically,

  $$P_2(x) \equiv \lim_{N_{total} \to \infty} \frac{N_2(x)}{N_{total}} \qquad \textit{Definition of probability} \qquad (1)$$

  Fig 5.1 above shows a schematic of the expected behavior of $P_2(x)$.

- Let's block hole #2. What would be the probability $P_1(x)$ that a bullet passing through the hole 1 arrives at the position x?

  $$P_1(x) = \lim_{N_{total} \to \infty} \frac{N_1(x)}{N_{total}} \qquad (2)$$

- When the two holes are open, what would be the probability P_{12} that a bullet, passing through either of the two holes 1 or 2 on the wall, will arrive at the position x on the screen?

  Classical particles typically behave in such a way that, it turns out, the probabilities are additive; that is,

  $$P_{12}(x) = P_1(x) + P_2(x) \qquad (3)$$

  Consequently, we affirm that this phenomenon (that deals with relatively large particles) **lacks interference**; that is to say,

  > *the opening or closing of one hole does not affect the individual probabilities associated with one aperture*.

  In other words, if a fraction $P_2(x)$ = 10% of the particles arrive at x when the aperture #1 is closed, then when the two apertures are open the fraction of the particles that arrive at x after passing through aperture 2 is still 10%.

5.1.B An experiment with light

A source emits monochromatic light waves, which pass first through a couple of slits, and then arrives at a screen. A detector on the screen measures the local intensity I (energy per unit time per unit area) of the incident light. We would like to understand the intensity distribution $I(x)$ established across the screen.

- When one of the slits is covered, a rather simple intensity distribution (either $I_1(x)$ or $I_2(x)$ respectively) is observed, as displayed in the Fig. 5.2 below. But when the two slits are open, a rather distinct wavy pattern $I_{12}(x)$ is observed. Definitely $I_{12}(x) \neq I_1(x) + I_2(x)$

  Consequently, we affirm that this phenomenon with light displays **interference** since the opening or closing of one slit does influence the effect produced by the other slit. In other words,

  one slit appear to behave different depending on whether the other slit is open or close.)

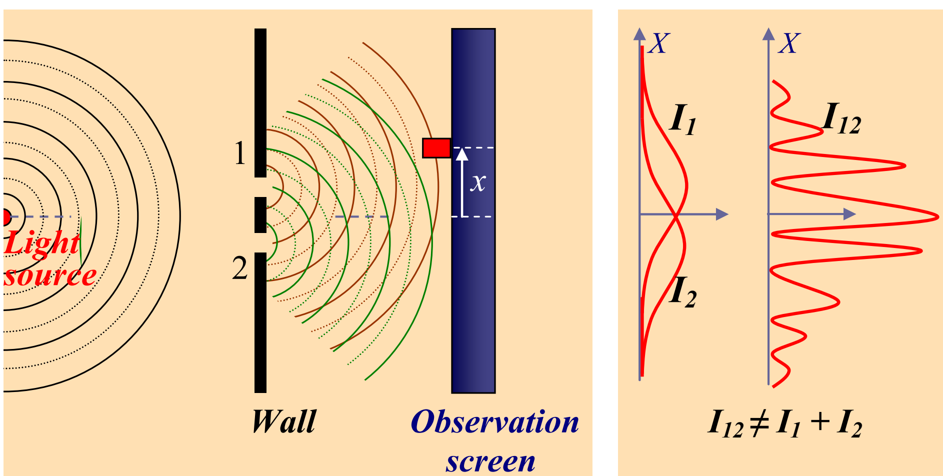


Fig. 5.2 An experiment similar to the one depicted in Fig. 5.1, but with light.

- **Wave interpretation**. We could attempt to explain this result by considering the interaction between the light beams that pass through the slit 1 and the slit 2. If the beams arrive at the observation screen in phase they will produce a bright spot; if they arrive out of phase a dark spot will result.

Notice, this plausible interpretation needs the participation of two entities (two beams).

- What about if, in addition, we introduce the **constituent granularity of light**? How does this new information affect our conclusion made in the paragraph above?

  In such a case, we can still argue that photons passing through the two different apertures would interfere at the screen.

  But, what about if we deem the intensity of the light source such that only one photon is emitted at a time, but, by employing a large exposure time, still collecting a lot of them at the detection screen?

  Based on the plausible wave interpretation stated above, The prediction is that the wavy pattern $I_{12}(x)$ would not form since "a single photon passing through a slit would have no other photon to interact with." As it turns out, this prediction is wrong.

- **This is what happens when only *one photon at a time* is emitted from the light source:**
  - ➢ If a photographic plate is placed on the screen, and the exposure time is chosen as short as to receive only one photon at a time, it is observed that:

    *Each photon produces a localized impac*t on the screen. (4)

    That is, a *single photon* does not produce on the screen a weak interference pattern like $I_{12}(x)$ (that is, a single photon does not spatially spread out); it produces just a single localized spot.

  - ➢ When the exposure time is drastically increased as to capture a large number of photon, a progressive formation of a wavy pattern $I_{12}(x)$ is observed*!*

    This was demonstrated (although in an experimental set up different than the one being described here) by G. I. Taylor in 1909, who photographed the diffraction pattern formed by the shadow of a needle, using a very weak source such that the exposure lasted for months.

It can be concluded then that interference does not occur between photons, but it is a property of single photons.

They arrive to the photographic plate in a statistical fashion and progressively build up a continuous-looking interference pattern. (5)

This has been confirmed in more recent experiments performed in 1989 by A. Aspect, P. Grangier, and G. Roger.

- **Why the pattern is affected so drastically when the two slits are open?**

  I speculate:

  A somewhat spatially-spread wavefunction Ψ_{in} is associated to an incident photon of wavelength λ.

  As the photon approaches the screen, the wavefunction passes not only through one slit but through the two of them. That is, Ψ_{in} is intersected by the two apertures. (The degree of Ψ_{in}'s spatial localization could be assumed to be of the order of λ. Consequently, a slit-separation of the order of λ would lead to more significant effects on Ψ_{in}.)

  Accordingly, the wavefunction at the other side of the screen carries the information about the two slits. That is to say, the initial wavefunction Ψ_{in} has been modified and became $\Psi_{through}$. The latter has the fingerprints of the two-slits.

  Sometimes the wavefunction of an incident photon is modified in one way and, accordingly, it statistically arrives at some place x on the screen; another incident photon is modified in potentially different way and arrives somewhere else x' ; overtime, the collection of many photons builds up the pattern $I_{12}(x)$.

  In short, there is not such a thing like a free single photon after the two slits; the wavefunction of such photon has already been modified by the two slits.

5.1.C An experiment with electrons

Let's consider an experiment in which mono-energetic electrons strike a wall having two slits; beyond the wall a screen full of detectors (only one is shown in the figure) allows recording the spatial pattern distribution of electrons-arrival along the screen.

The arrival of an electron at the detectors is sensed as a detector's 'click'.

Electrons arrive at the detector in identical pieces (one 'click' per electron).

When one electron at a time is emitted by the source, only one detector's 'click' is detected at a time.

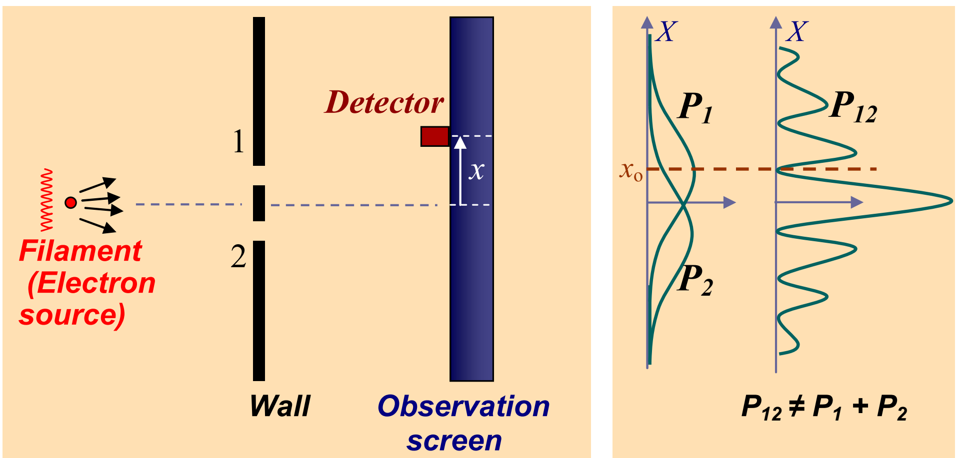


Fig. 5.3 *Schematic of the two-slit experiment using electrons.*

The results sketched in the diagram above indicate that,

electrons behave like light. (6)

Indeed interference pattern due to incoming single electrons was observed in 1989 by A. Tonomura, J. Endo, T. Masuda, T. Kawasaki, and H. Ezawa

The observed interference effect in the detection of electrons and photons (as described above) indicate that the following statement (plausible in the classical world) is not true:

"Each electron either goes through aperture 1 or through the aperture 2" (7)

For:

- When we close aperture 1 (so we know for sure the electron passed by the aperture 2) the detector at $x=x_o$ collects a given number of electrons, let's say, $N_2(x_o)$= 348.

- When we close aperture 2 (so we know for sure the electron passed by aperture 1) the detector at $x=x_o$ collects a given number of electrons $N_1(x_o)$= 1965
- How is that when we leave both apertures open the detector at $x=x_o$ collects *none*?

The wavy pattern $P_{12}(x)$ (as shown in the Fig. 5.3 above) is then not compatible with the statement (7). When the two apertures are open, we do not know which aperture a specific electron passed through.

This type of uncertainty is rooted in the quantum behavior of atomic objects. Further, it appears that Quantum Mechanics protects this uncertainty as to defeat any attempt to overcome it. In the next section we will describe those attempts in a more formal detail.

5.1.D Attempts to track the trajectory of the electrons' motion

Watching electrons with a light source.

As shown in Fig. 5.4 below, a light source has been added to the previous set up in an attempt to watch what specific aperture electrons choose to pass through.

> **Procedure:** When an electron passes aperture-1, for example, we should observe photons being scattered from a region nearby the aperture 1. These electrons (type 1) will be tabulated under the table-column P_1'
>
> Similarly, for electrons scattering photons from a region near the aperture 2 a second tabulated column will be generated.

When this experiment is performed, the result, it turns out, is:

Result: $P_{12}' = P_1' + P_2'$

Just as we expect for classical particles. (That is, the interference pattern, obtained when we do not watch the electrons, does not occur any more.)

Conclusion

"When we watch the electrons, they behave different than when we do not use light to watch them." (8)

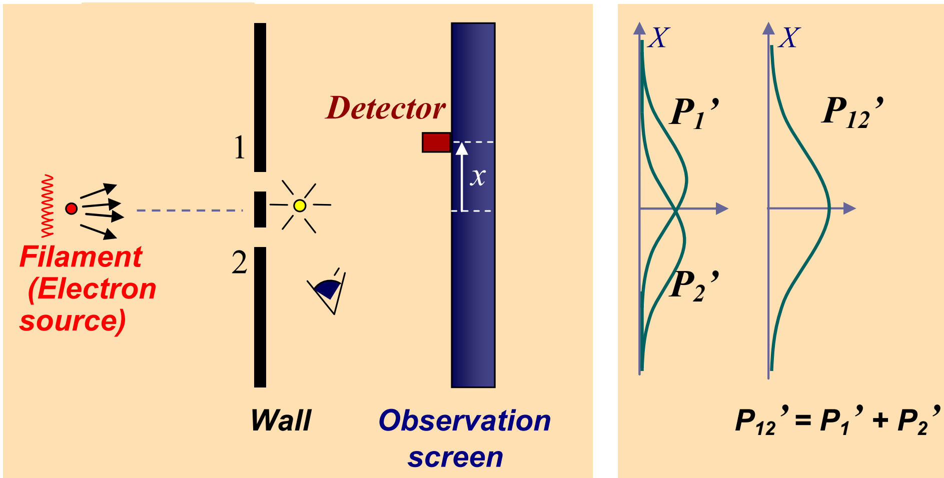


Fig. 5.4 *Schematic of the two-slit experiment using electrons while watching them.*

What about sing more gentle photons?

Since, as we know, light is an electromagnetic entity (which interact with charged particles), probably we should expect electrons to behave different when using photons to watch them.

- The motion of the delicate electrons may be perturbed by the linear momentum $p = h/\lambda$ carried by the photons.
- An alternative would be to use more gentle photons; those of longer and longer wavelength.

We will encounter a limitation in this approach, however. As we know, when using light there is a restriction on the ability to distinguish how close two spots are apart from each other and still be seen as two separate spots: If the separation between the objects is smaller than the wavelength of the radiation being used, we will not able to distinguish them.

Thus, in our two-slit experiment with electrons,

- as we use longer and longer wavelengths (as to minimize the perturbation on the electron's motion) our ability to distinguish the slits will be worst and worst. We will not be able, then, to 'see' which slit a given electron passed through.

The above observation helps illustrate the unavoidable **perturbation role** played by **a measurement**. (We knew this, but we did not expect to be so drastic with small particles like electrons.)

It is impossible to arrange the light in such a way that one can tell which hole the electron went through and at the same time not to disturb the pattern. (9)

5.2 The Heisenberg's Uncertainty Principle

It was suggested by Heisenberg that the new law of Nature could only be consistent if there were some basic limitations in our experimental capabilities not previously recognized. Two important cases will be discussed with some detail in the sections below; one pertaining the position and momentum variables, the other Energy and time (indeed, the uncertainty principle involves a pair of variables.)

5.2.A Uncertainty in the position Δx and linear momentum Δp.

Originally Heisenberg stated:

If you make the measurement of any object, and you can determine the x-component of its momentum with an uncertainty Δp*, you cannot at the same time, know its* x*-position more accurately than* $\Delta x = h / \Delta p$ (10)

Let's consider a specific example to see the reasons why there is an uncertainty in the position and the momentum if quantum mechanics is right.

Example: Particles passing through a single slit.
We will focus our interest in the vertical component of the particles' linear momentum, p_y.

As shown in Fig. 5.5 , since the particles come from a source far away from the slit, they arrive with a linear momentum oriented practically horizontal. Thus,

Before arriving to the slit: $p_y = 0$.

However, we do not know with certainty the vertical positions of the particles $\Delta y = \infty$.

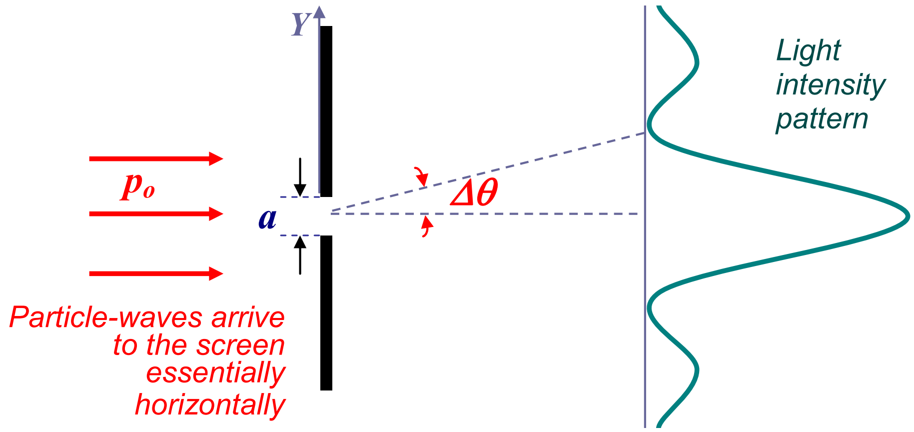


Fig.5.5 *Diffraction of particles passing through a slit.*

Let' use an aperture of size "*a"* as an attempt to localize better the vertical position of some of the particles contained in the incident beam. That way,

after the particles had passed through the aperture we would know the y-position with a precision of $\Delta y = a$. (11)

However, notice that for the particles passing through the slit we lose information about their vertical momentum (it was Δy=0). The reason is that, according to the wave theory, there is a certain probability that the particles will deviate from the straight incoming direction, and rather form a spread diffraction pattern. In other words,

there is a probability that the passing particles acquire a vertical momentum.

A measurement of the angular spread is given by the angle $\Delta\theta$ at which the diffraction pattern displays a minimum (see the inclined dashed-line in the figure above). From the diffraction theory, the condition for the occurrence of that minimum is expressed by,[2]

$$a \ \Delta\theta = \lambda$$

A corresponding measurement of the spread of the vertical momentum values is then no better than $p_o\Delta\theta$; *i. e.* ,

$$\Delta p_y \geq p_o \ \Delta\theta = p_o\lambda / a \qquad (12)$$

Notice, if a smaller value of "a" were chosen (in order to get a better precision of the particles' vertical position) a corresponding larger spread of Δp_y is obtained.

Introducing the quantum mechanics concept that the momentum p_o is given by $p_o = h/\lambda$, (11) and (12) imply

$$\Delta p_y \ \Delta y \geq h \tag{12}$$

This expression indicates, the better precision Δy in the particle's vertical position (using a narrower the slit) the greater the spread in the vertical linear momentum (the greater the probability that a passing particle ends up having a very large vertical linear momentum).

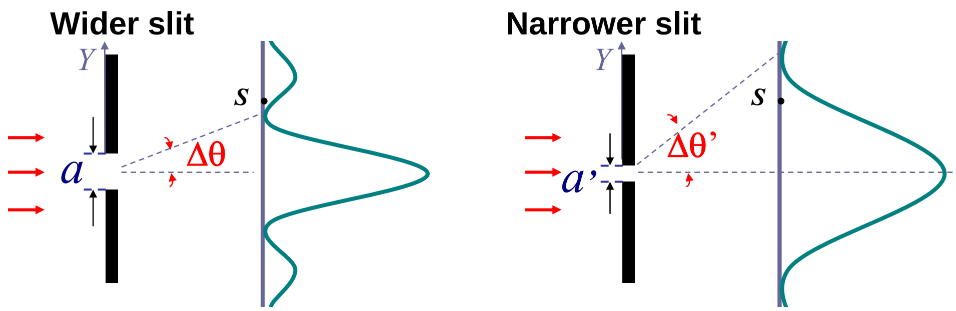


Fig. 5.6 *The probability that a particle ends up at S increases as the slits gets narrower*

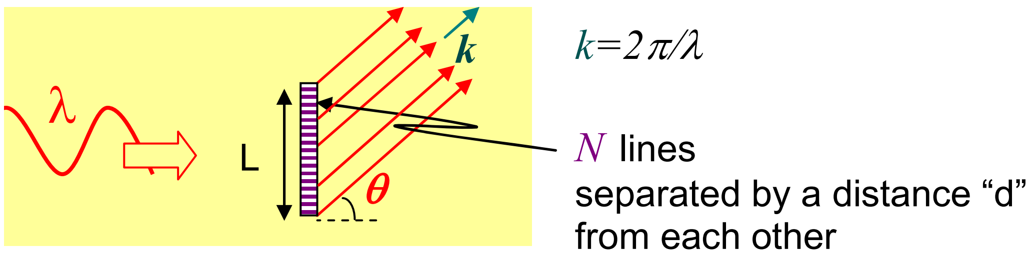
5.2.B Relationship between the uncertainty of the energy content ΔE of a pulse and the time Δt required for the measurement.

In this section, the working principle of a grating is used to illustrate the energy-time uncertainty principle.

- First, we familiarize with the functioning of a grating, and emphasize its inherent limited spectral resolution (inability to distinguish waves of slightly different frequency,) which in principle has nothing to do with quantum mechanics).
- Then, by associating the radiation frequency (used in the description in the functioning of the grating) with the energy of the radiation, as well as properly interpreting the time needed to make

5.2.B.a Gratings

Let's consider a fully illuminated grating of length L and composed of N lines equally separated by a distance "d"



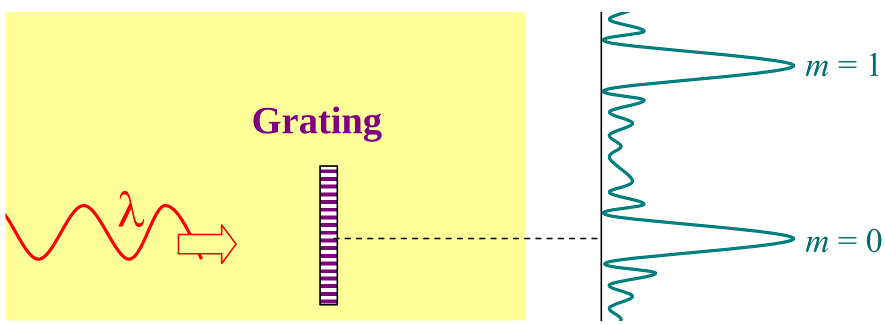


Fig.5.7 *For radiation of a given wavelength, several maxima of intensity (m=0, 1, 2,…) are observed at different angular positions as a result of constructive interference by the N lines sources from the grating.*

Calculation of the maxima of interference by the method of phasors[3]

- **Case: Phasor addition of two waves**

  Wave-1 = A Cos(kx) → phasor-1 = $\Psi_1 = Ae^{i[kx]}$

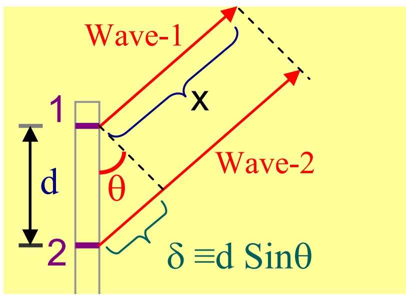
  Wave-2 = A Cos(kx+kδ) → phasor-2 = $\Psi_2 = Ae^{i[kx+k\delta]}$

  $A\,\mathrm{Cos}(kx) = \mathrm{Real}(\Psi_1)$

  $A\,\mathrm{Cos}(kx+k\delta) = \mathrm{Real}(\Psi_2)$

Notice,

A path difference of	gives a	phase difference of
δ	$\longrightarrow$	$k\,\delta = \frac{2\pi}{\lambda}\delta$
λ	$\longrightarrow$	$k\,\delta = 2\pi$



$$\Psi_1 = Ae^{\,i[\,kx\,]}$$ (kx: phase)

$$\Psi_2 = Ae^{\,i[\,k\,(x+\delta)\,]} = e^{\,i[\,k\,x + k\delta\,]}$$

Fig. 5.8 *Real-variable waves are represented by their corresponding complex-variable phasors.*

$$\Psi = \Psi_1 + \Psi_2 = Ae^{\,i[\,kx\,]} + Ae^{\,i[\,k\,x + k\delta\,]}$$

$$A\,\mathrm{Cos}(kx) + A\,\mathrm{Cos}(kx+k\delta) = \mathrm{Real}(\Psi_1 + \Psi_2)$$
$$= \mathrm{Real}(\Psi)$$

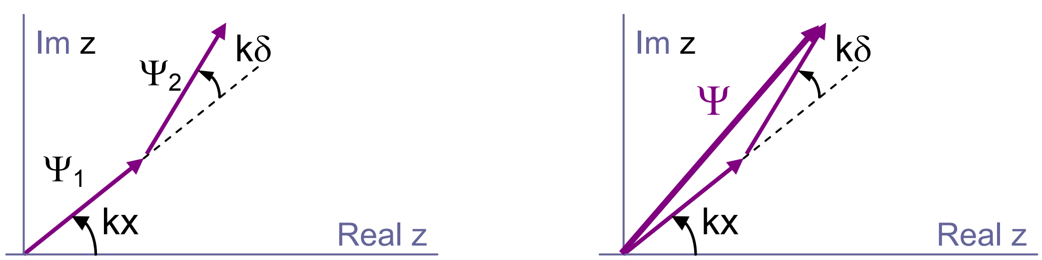


Fig. 5.9 *Addition of two phasors in the complex plane.*

- **Case: Phasor addition of N waves**

$$\Psi = \Psi_1 + \Psi_2 + \ldots \Psi_n =$$

$$= Ae^{\,ikx} + Ae^{\,i(\,kx+k\delta\,)} + Ae^{\,i(\,kx+2k\delta\,)} + \ldots + Ae^{\,i(\,kx+(n-1)k\delta\,)}$$

$$= e^{i\,kx}\left[A + Ae^{ik\delta} + Ae^{i2k\delta} + \ldots + Ae^{i(N-1)k\delta}\right]$$

$$\Psi = e^{i\,kx}\left[\sum_{s=0}^{N-1} Ae^{i\,s\,k\delta}\right] \qquad (13)$$

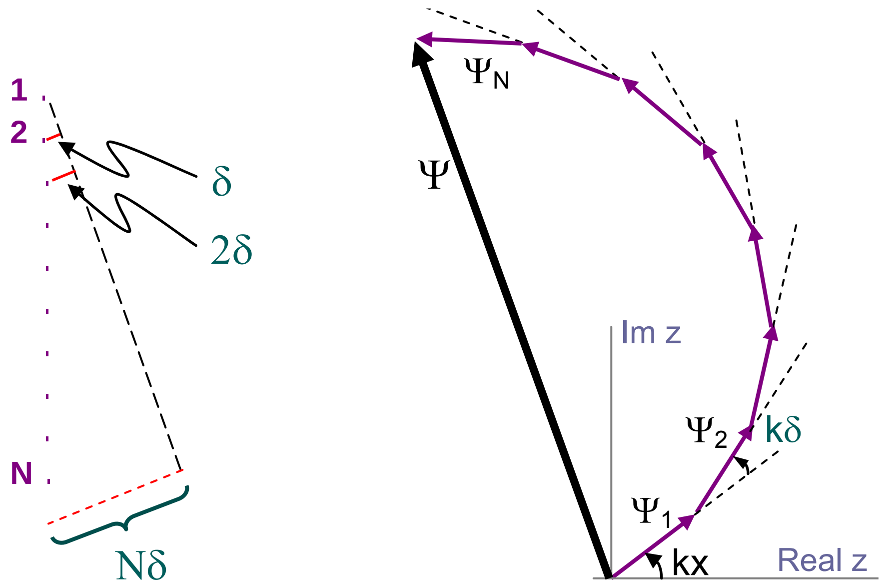


Fig. 5.10 *Addition of N phasors in the complex plane.*

Condition for constructive interference

This occurs when contiguous interfering waves travel a path difference δ exactly equal to an integer number of wavelengths

$$\delta = m\,\lambda \qquad (m = 0,1,2,3,\ldots); \qquad (14)$$

which take place at the angles $\theta = \theta_m$ that satisfies:

$$\delta \equiv d \sin\theta_m = m\,\lambda$$

Equivalently, since $k=2\pi/\lambda$, the max of constructive interference occurs when the phase difference $k\delta$ between two contiguous waves is exactly equal to an integer number of 2π;

$$k\,\delta = m\,2\pi \qquad (m = 0,1,2,3,\ldots).$$

Condition for max of constructive interference

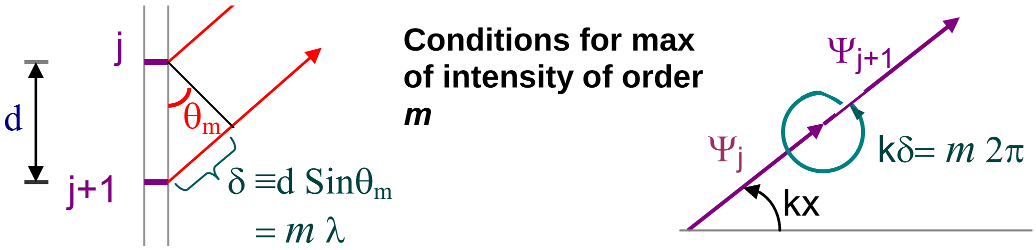


Fig. 5.11 Condition in which the phase difference between two contiguous phasors is m 2π. They contribute to the max of order m (see Fig. 5.12 below).

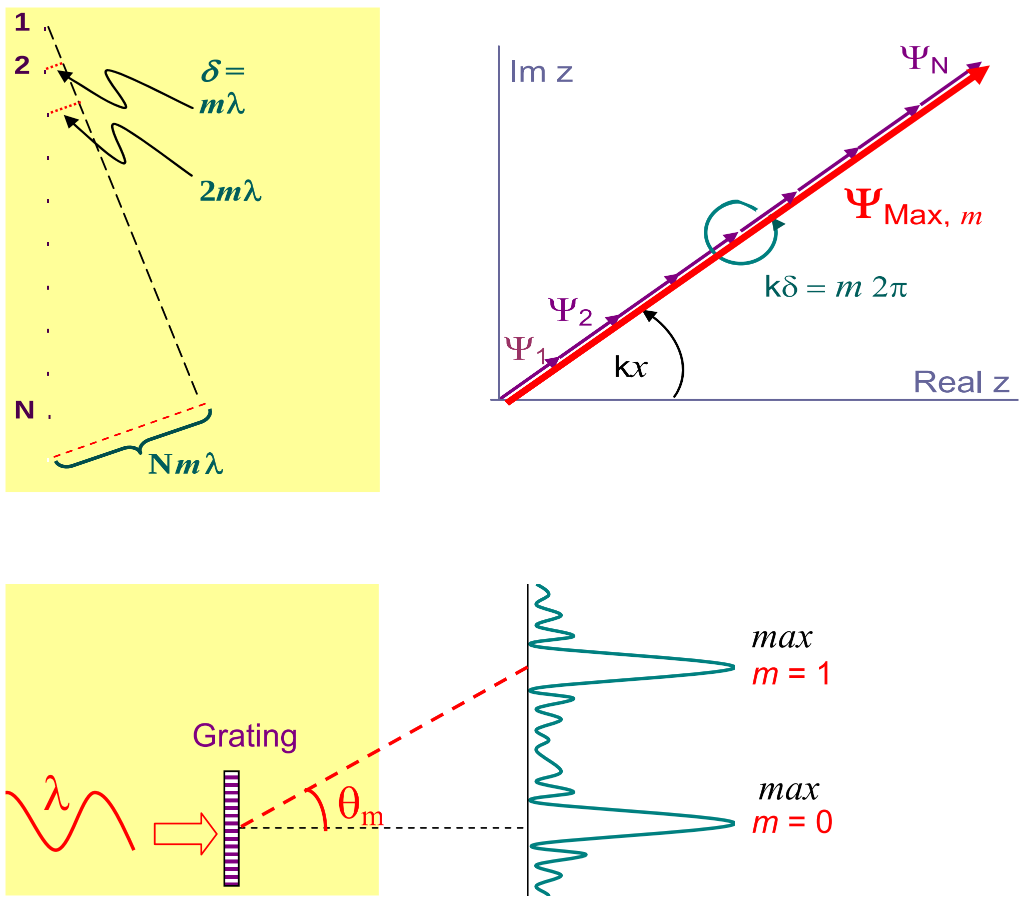


Fig. 5.11 *Condition to produce a maximum of intensity (of order m).* ***Top:*** *All the N phasors add up as to give a phasor of the largest possible amplitude. Bottom: Visualization of the corresponding maxima of order m (m=0, 1, 2, …).*

Condition for evaluating the sharpness of the zero-th order peak.

Around a peak of maximum intensity, the intensity drops to zero, thus defining a line thickness. We would like to calculate the angular broadening associated to that thickness of line intensity.

Lets consider first the max of order $m=0$. Using the method of phasors, we realize that

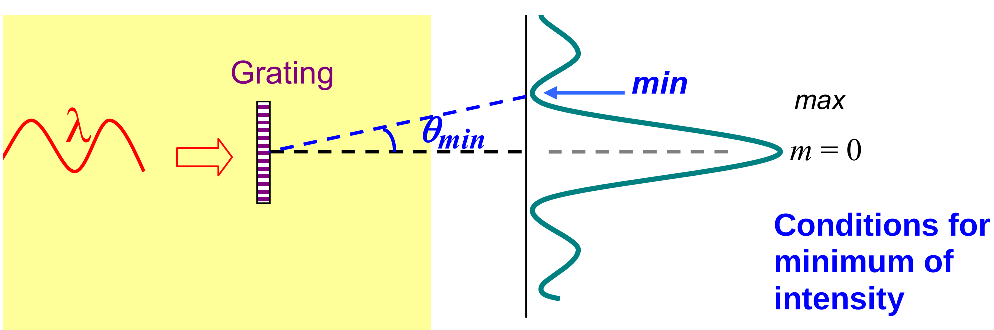
A path difference of between contiguous radiators	gives a	phase difference of
δ	$\longrightarrow$	$k\,\delta = \frac{2\pi}{\lambda}\delta$
λ/N	$\longrightarrow$	$2\pi/N$
For N oscillators		the total accumulate phase would be 2π;

That is, a zero amplitude wave would be obtained if

$$\delta = \lambda / N \tag{15}$$

Since $\delta \equiv d \sin(\theta)$, the angle θ_{min} that locates the minimum of intensity (next to the maximum of order $m=0$) is given by,

$$N\, d \sin(\theta_{min}) = \lambda \tag{16}$$



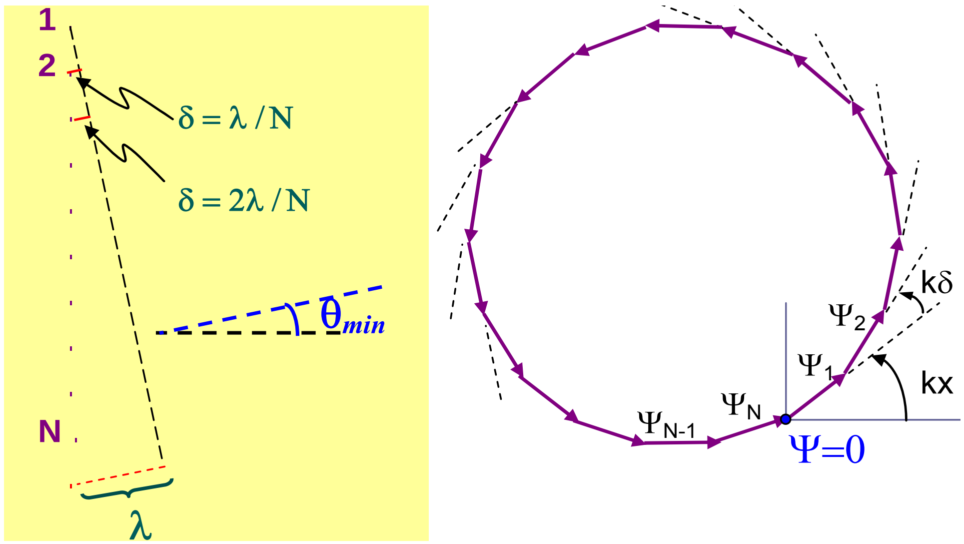
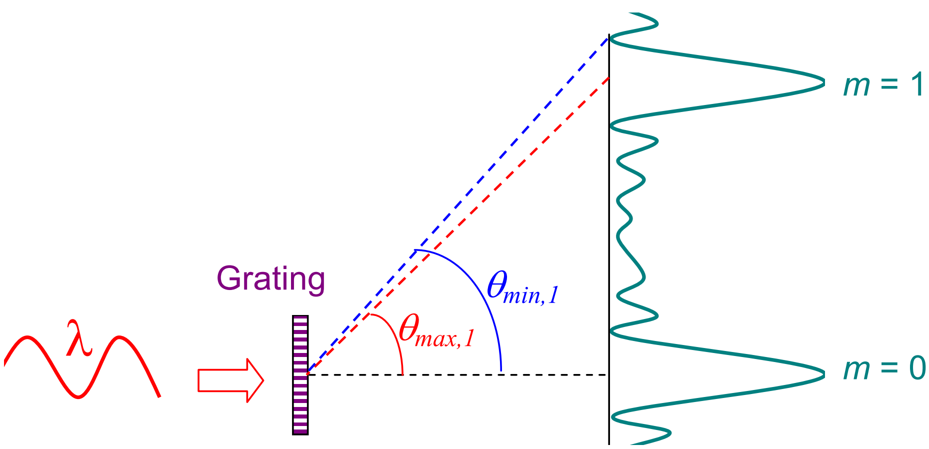


Fig. 5.12 Condition for finding the angular position of the minimum of intensity closest to the maximum of order m=0 (hence, giving a measurement of the angular sharpness of the peak of order m=0).

Condition for evaluating the sharpness of the peak of *m*-th order maximum.

The angular location of the minima of intensity around the *m*-th order maximum intensity can be calculated in a similar fashion.



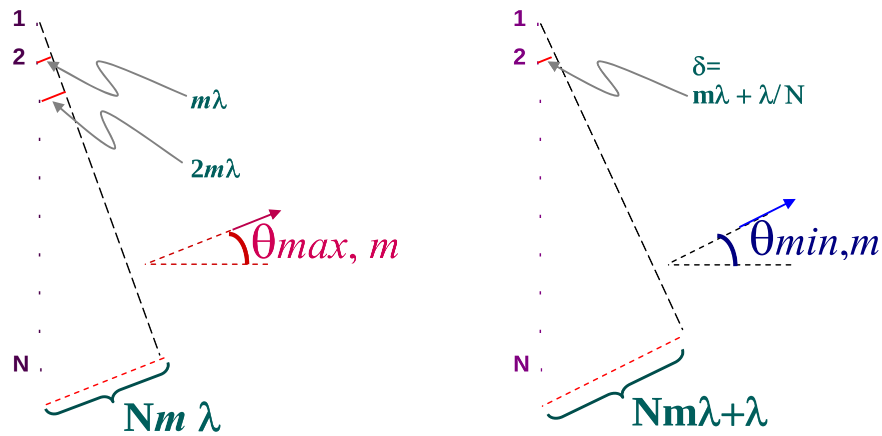


Fig.5.13 *The diagrams (top and bottom) show the angle $\theta=\theta_{max;\ m}$ that determines the intensity maximum of m-th order. ($\delta \equiv d\ Sin\theta\ = m\ \lambda$). As the angle increases a bit, a minimum of intensity will be reached, which occurs at the angle $\theta=\theta_{min;\ m}$, which is determined by the condition $\delta \equiv d\ Sin\theta = m\ \lambda + \lambda/N$ (see also Fig 5.14 below.).*

To obtain a minimum of intensity we choose a path difference between contiguous waves equal to δ = $m\lambda$ + λ/ N , which gives a corresponding phasor rotation kδ=$m2\pi$ + 2π/ N. Notice, however, that the latter rotation is equivalent to a net rotation of 2π / N (since a rotation of $m2\pi$ brings the phasor to the same place.) Thus we are choosing a net phase difference between two contiguous waves equal to 2π /N. After N rotation we will have completed a full rotation that brings the total phasor to zero. This is illustrated in Fig. 14.

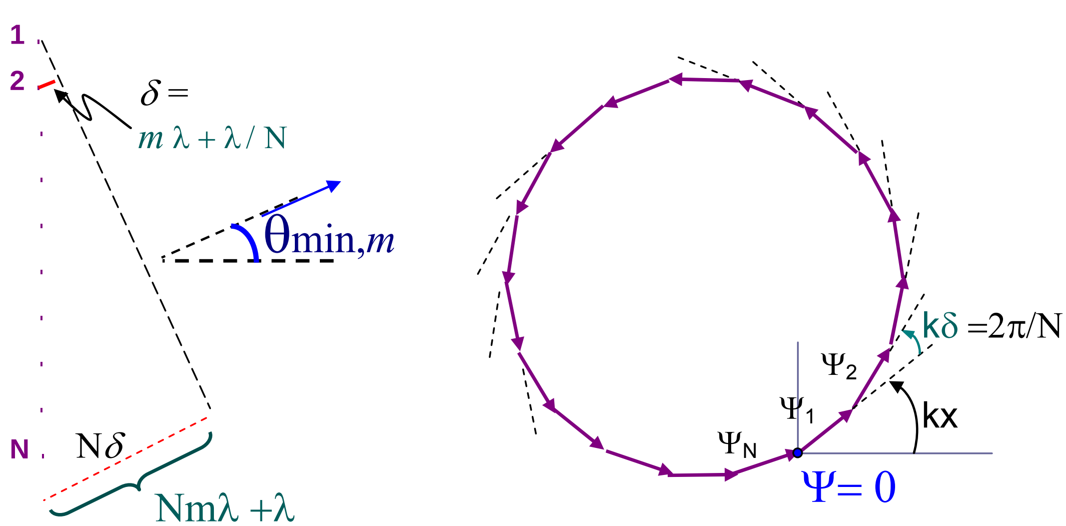


Fig.5.14 *The diagram shows the condition that determine the angle* $\theta=\theta_{min;\,m}$ *(closest to the angle* $\theta=\theta_{max;m}$*) for which the intensity is zero* $\delta \equiv d\ Sin\theta = m\ \lambda + \lambda/N$

$$\delta \equiv d \sin(\theta_{max,\,m}) = m\,\lambda \qquad (17)$$

Gives the angular location of the max of order m.

$$\delta = m\,\lambda + \lambda/N \qquad (18)$$

$$N\delta \equiv Nd \sin(\theta_{min,\,m}) = mN\lambda + \lambda$$

This expression gives the angular position that locates the minimum immediately next to the max of order m.

5.2.B.b Gratings and Spectral Resolution

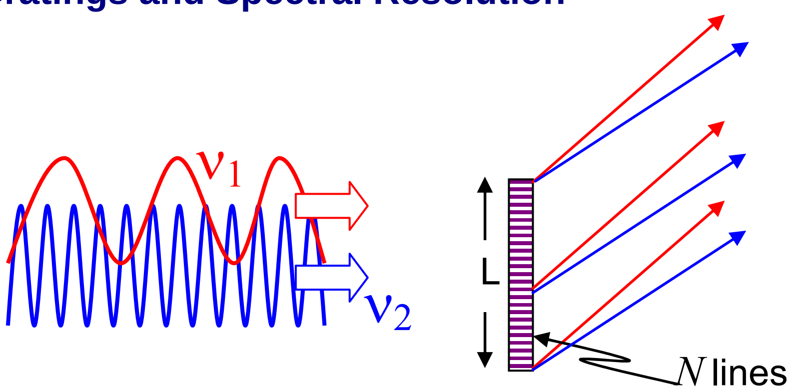


Fig.5.15 *Radiation of different wavelength (or frequency) produces maxima of intensity a different angular positions*

Given λ_1, and working around the maximum of the m-th order, the following will happen:

- The grating will be able to distinguish the peaks formed by λ_1 and by another wavelength λ_2 if the following condition is satisfied

$$Nm\lambda_1 - Nm\lambda_2 \geq \lambda_2 \qquad (19)$$

The interpretation of this condition is that the two peaks of intensity (one corresponding to λ_1 and the other to λ_2) are separated by several small maxima and minima of intensity).

- The closer the value of λ_2 gets to the value of λ_1, (assuming $\lambda_2 < \lambda_1$) the more restringing the condition becomes. At most, λ_2 can be allowed to grow up to fulfill the following condition,

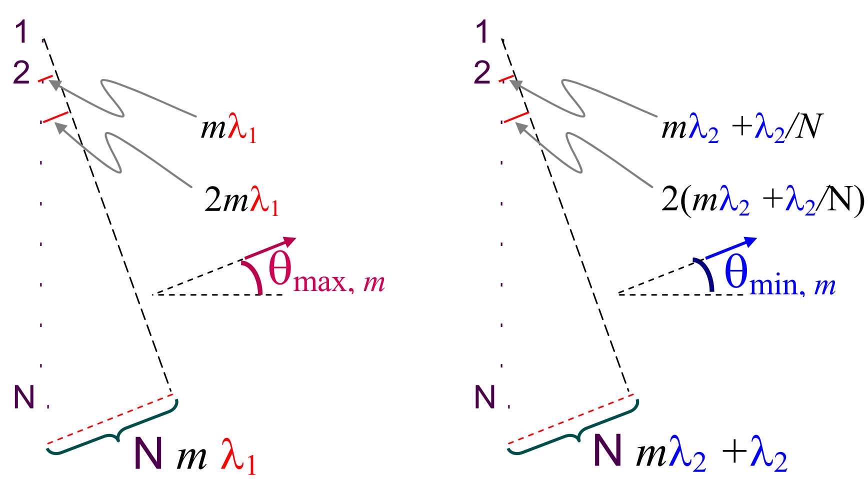
$$Nm\lambda_1 - Nm\lambda_2 = \lambda_2 \qquad (20)$$

Compare the condition (20) with expression (17) and (18). Indeed, if we write the last expression as

$$Nm\lambda_1 = Nm\lambda_2 + \lambda_2 ,$$

we notice that the left side correspond to the condition of maximum of intensity produced by the wavelength λ_1 and the right side establishes the condition of minimum of intensity for the wavelength λ_2 (see figure 5.16 below.)

In such a case the two maxima of intensity (corresponding to each wavelength) have become so close that the minimum of intensity of the smaller wavelength (λ_2) overlaps with the λ_1's maximum of intensity. This is the Raleigh's criterion of resolution.



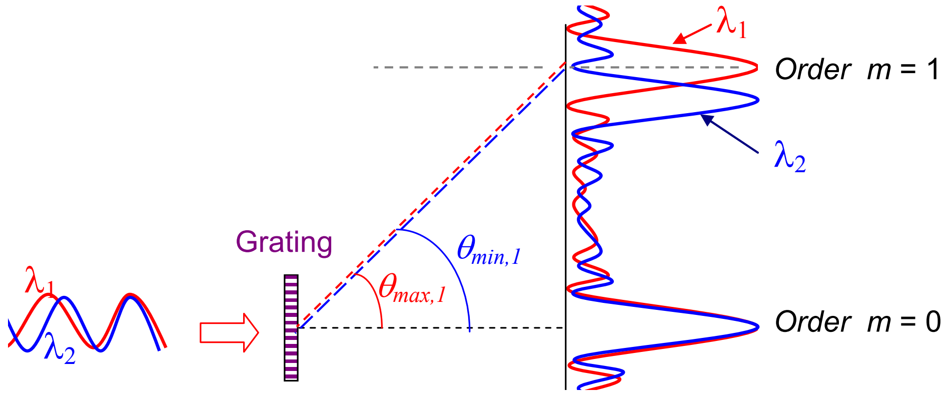


Fig. 5.16 Raleigh's criterion of resolution illustrated for the case m=1

Expression (19) can be written as $Nm(\lambda_1-\lambda_2) \geq \lambda_2$, or $Nm(\Delta\lambda) \geq \lambda_2$, thus indicating that,

given a grating of N lines, and while operating at a wavelength λ, two wavelength can be distinguished (according to the Raleigh criteria) if their wavelength difference $\Delta\lambda$ satisfies (21)

$$\Delta\lambda \geq \frac{\lambda}{N\,m}$$

Equivalently, to distinguish the presence of two waves whose wavelength difference is $\Delta\lambda$, requires a grating of the following number of lines

$$\mathrm{N} \geq \frac{1}{m}\frac{\lambda}{\Delta\lambda}$$ *(when using the maxima of m-th order)* (22)

If the separation between the lines in the grating is "d", the expression above can also be written as,

$$\frac{L}{d} \geq \frac{1}{m}\frac{\lambda}{\Delta\lambda}$$ Here L is the length of the grating illuminated by the incident radiation

Since $\frac{\Delta\lambda}{\lambda} = -\frac{\Delta\nu}{\nu}$, this expression can be written also as

$$\Delta\nu \geq \frac{\nu}{mN} \equiv (\Delta\nu)_{\min} \tag{23}$$

This expression gives the minimum frequency-difference between two waves that a grating of N lines is able to distinguish as, in fact, two distinct incident frequencies (when working around the maximum of order m).

5.2.B.c Condition for the minimum length time Δt required for the measurement of the energy E with a resolution ΔE.

- Notice in Fig. 5.17 below that,

  for a given $\lambda = \lambda_1$ (or equivalently a given frequency $\nu = \nu_1$) and for a given angular orientation θ_m at which the grating produces an order-m maximum of intensity,

  the time T_{AB} needed by the light to travel the distance AB is,

$$T_{AB} = \frac{AB}{c} = \frac{\mathrm{N}m\lambda}{c} = \frac{\mathrm{N}m}{\nu}$$

Using (23), the expression above for T_{AB} can also be placed in terms of $\Delta\nu$, the smallest frequency difference the grating can resolve

$$T_{AB} = \frac{1}{\nu / \mathrm{N}m} = \frac{1}{\Delta\nu_{\min}} \tag{24}$$

- Notice in the figure below that, in order to resolve the frequency content of the incident radiation with a resolution $\Delta\nu$, the latter has to last at least a time $(\Delta t)_{measurement}$ = T_{AB}=1/$\Delta\nu$. That is,

$$(\Delta t)_{measurement} \geq \frac{1}{\Delta\nu} \tag{25}$$

Otherwise, if the incident radiation lasted shorter then it wouldn't be possible to form the interference pattern (no all the N components would participate).

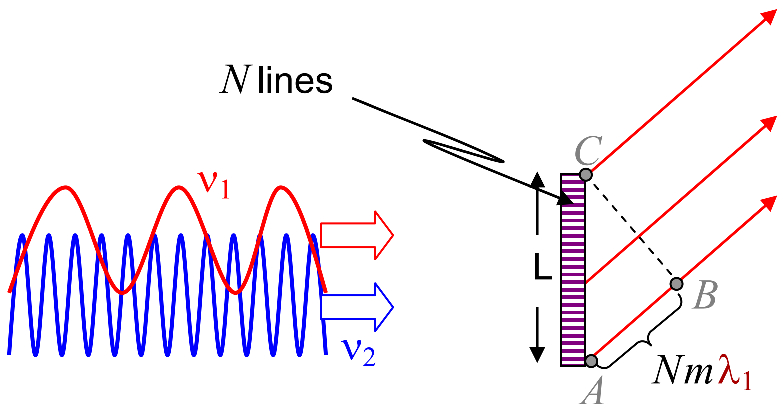


Fig. 5.17 Aiming to distinguish the two frequencies ν_1 and ν_2 ($\Delta\nu = \nu_2 - \nu_1$) a setup to detect the maximum of order m, is chosen. Such constructive interference requires the simultaneous participation of the N wave components in the interfering wavefront CB. The formation of such wavefront CB requires a pulse duration Δt of at least equal to the time light requires to travel the distance AB, Δt = AB/c. It turns out AB/c = 1/ $\Delta\nu$.[4]

We have arrived to a very interesting interpretation:

In order to find out whether an incident radiation contains harmonic wave components whose frequencies differ by $\Delta\nu$, the minimum length of the pulse has to be $1/\Delta\nu$. (26)

The less uncertainty we want to have concerning the frequency components in the pulse, the longer pulse we need (i.e. a longer measurement-time will be required.)

We would need more (measurement) time if we want to find out the spectral content with less uncertainty.

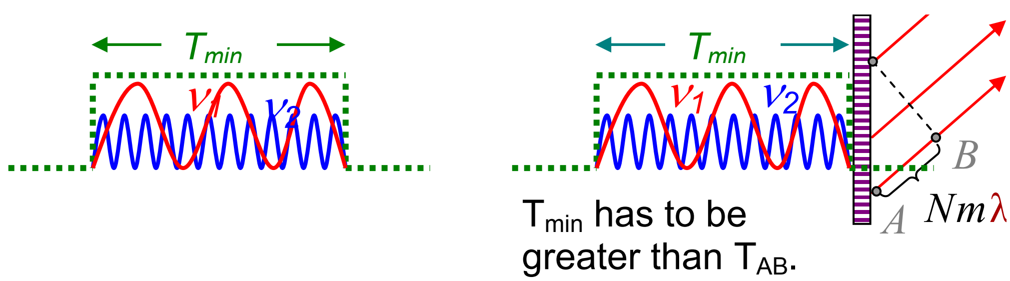


Fig. 5.18 There is a close relationship between the precision $\Delta\nu$ with which we want to know the spectral content of a pulse, and the minimum time duration T that the pulse is required to have in order a measurement with such spectral precision: $T_{min}=1/\Delta\nu$.

This is also illustrated in the Fig. 5.19 where it is assumed to know the frequency ν_o is contained in the pulse f.

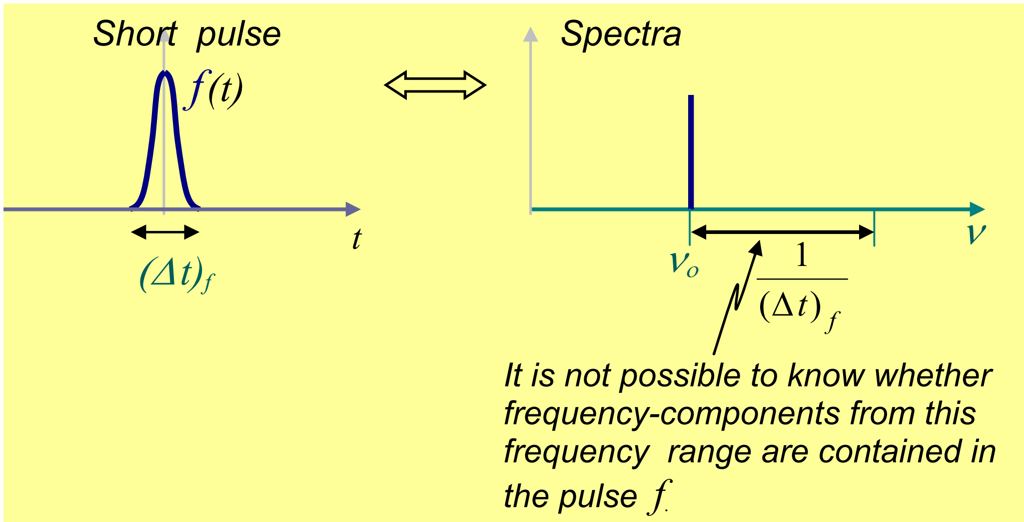


Fig. 5.19 The shorter the pulse $(\Delta t)_f$, the larger the uncertainty in knowing the frequency content of the pulse. This occurs because of the short time $(\Delta t)_f$ available for the measurement.

Notice, the results just described have nothing to do with quantum mechanics. It is a property of any wave.

It occurs, however, that quantum mechanics associate a wave character to the particles; hence the variables associated to the wave-particle (momentum, position, energy) become subjected to the frequency/time of position/momentum uncertainties.

For example, applying the concept of quantum mechanics $E = h\nu$, expression (25) can be expressed as,

$$(\Delta E)(\Delta t)_{measurement} \geq h \tag{27}$$

5.2.C. Uncertainty principle and the resolving power of a microscope.

5.2.C.a Image formation and the resolving power of a lens

- P focuses at T

  A condition for a clear focused image formation is that rays take an equal time to travel from the source point to the image point.

- A point P’ will focus at a region very close to T.

  If P’ is too close to P, their image will superimpose and become too blur to distinguish them; they would appear to be the same point.

  How close is too close?

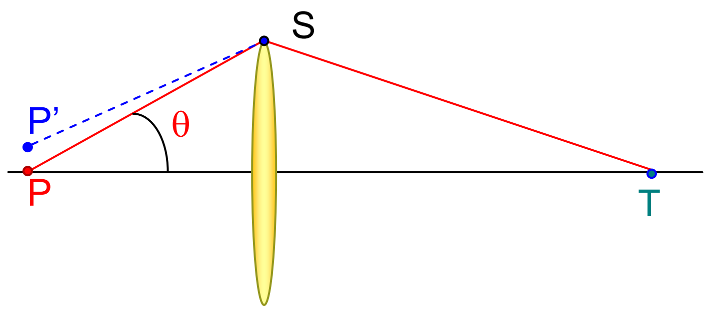


Fig. 5.20 *A point P is imaged by the lens on point T. How far apart have to be a point P’ so that the lens clearly image it at a point different than T?*

- The condition that point P’ focuses at a different point than T is that

  t(P’ST) and *t*(PST) have to be different.

  (otherwise they would focus at the same point.)

  Again, how much different do *t*(P’ST) and *t*(PST) have to be so that we can conclude that these two paths do not belong to a set of rays forming a clear image?

- **The condition of resolution**

  Starting with the path PST and its corresponding time t(PST), let’s calculate the time *t*(P’ST) as the point P’ moves out of axis away from P.

  The condition of resolution (the ability to distinguish P and P’) states that

  $t(P'ST) - t(PST) >$ one period of oscillation of the radiation being used for imaging (28)

  $t(P'ST) - t(PST) > 1/\nu$

Using the diagram in Fig. 5.21, and in terms of *d* and θ, the expression above becomes,

$$[\ t(\mathrm{P'ST}) - t(\mathrm{PST})\ ] = d \sin\theta / \mathrm{v} = d \sin\theta / (\mathrm{c}/n)\,,$$

where n is the index of refraction of the medium

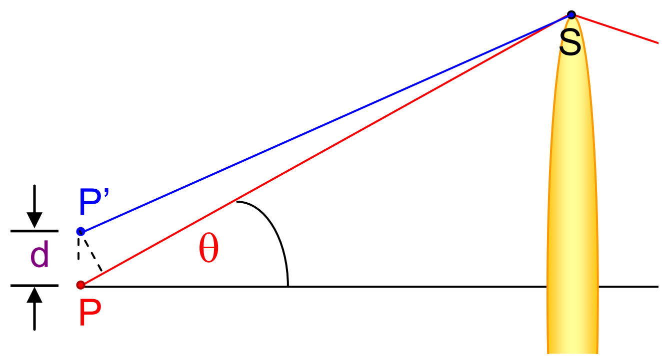


Fig. 5.21 *Diagram to calculate the difference in traveling time by two rays.*

Thus, according to (28), the condition of resolution can be expressed as,

$$d \sin\theta / (\mathrm{c/n}) \;>\; \frac{1}{v}$$

Or,

$$d \;>\; \frac{\lambda}{n \sin\theta} \qquad \textit{Resolving power of a lens} \qquad (29)$$

Thus, $\lambda / n \sin\theta$ is the minimum distance that two points P and P' need to be separated in order to be imaged as two different points.

The quantity in the denominator is defined as the numeral aperture of the lens,

$$\mathrm{NA} \equiv n \sin\theta \qquad (30)$$

and the resolving power is typically expressed as

$$\textit{Resolving power } R = \frac{\lambda}{NA} \qquad (31)$$

5.2.C.b Watching electrons through the microscope

We wish to measure as accurate as possible the position of an electron. We have to take into account, however, that the very act of observing the electron with photons disturbs the electron's motion.

> The moment a photon interacts with an electron, it recoils in a way that cannot be completely determined (there will be an uncertainty in the possible 'exact' recoil direction.)

If we detect a signal through the microscope, it would because the photon of wavelength λ (and of linear momentum $p=h/\lambda$) has recoiled anywhere within the lens angle of view 2θ'. That is, the x-component of the photon 's momentum can be any value between $p\text{Sin}\theta$ and $-p\text{Sin}\theta$. $(\Delta p_x)_{photon} = 2p\,\text{Sin}\theta$. By conservation of linear momentum, the electron must have the same uncertainty in its x-component linear momentum $(\Delta p_x)_{electron} = (\Delta p_x)_{photon}$. Thus, the uncertainty in the x-component of the recoiled electron's linear momentum is

$$(\Delta p_x)_{electron} = 2p\,\text{Sin}\boldsymbol{\theta} = 2(h/\lambda)\text{Sin}\boldsymbol{\theta} \qquad (32)$$

But, where is the electron (after the interaction with the photon)? How accurate can we determine its position. At the moment of detection of the recoiled photon the electron must have been somewhere inside the focused region of the lens. But the size of that region, according to expression (29), is not smaller than

$$\Delta x = \lambda / Sin\theta \qquad (33)$$

The product of the two uncertainties gives,

$$\Delta p_x \Delta x = 2h \qquad (34)$$

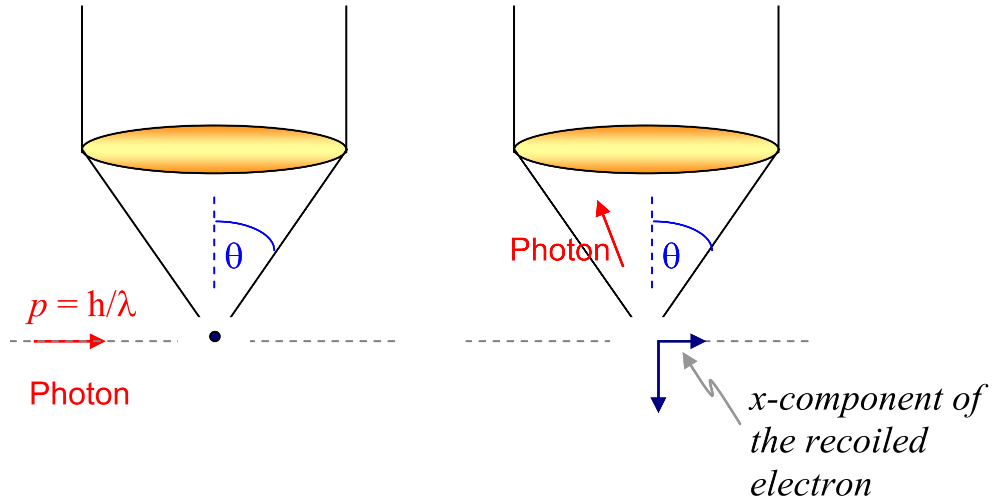


Fig. 5.21 The Numerical aperture nSinθ of the lens determines the uncertainty $\Delta p_x = 2pSin\theta$ with which we can know the electron's momentum.

QUESTION;

QA A light pulse of length duration $(\Delta t)_1$ has wave components of energy within the range $(\Delta E)_1$.
The energy content can be revealed by making the pulse to pass through a spectrometer; let's call this energy-content (or pulse fingerprint) "Spectrum-1"

A similar pulse passes first through a slit. But the slit is opened only a fraction of time $(\Delta t)_1/10$. (That is, an experimental attempt is made to reduce the temporal duration of the pulse.)
When the resulting pulse passes through a spectrometer, how different the new experiment would be compared to "Spectrum-1"?

A possible answer would be: Since the pulse duration has been reduced, the energy content should have been spread (according to the Heisenberg's uncertainty principle).
Question: If this answer is true, why new wave components of correspondingly different energy contents (compared to the original pulse) would show up in the new pulse (and revealed in the new spectrum)? Those new component were not there in the original pulse.

While, mathematically, a thinner pulse has different Fourier decomposition than a thicker pulse, how can we understand this physically?

Another way to post the question is: If a red color light-pulse passes through a slit that opens only an infinitesimal length of time, could we eventually be able to make this pulse to become a white-light pulse?

---

[1] Richard Feynman, "The Feynman Lecture on Physics," Vol-I, Chapter 37, or Vol III Chapter 1; Addison-Wesley, 1963.

[2] In the following section (when dealing with the functioning of a grating) we will justify this condition in more detail

[3] Richard Feynman, "The Feynman Lecture on Physics," Vol-I, Chapter 30, or Vol III Chapter 1; Addison-Wesley, 1963.

[4] Richard Feynman, "The Feynman Lecture on Physics," Vol-I, Section 38-2., Addison-Wesley, 1963.